

Lagrange Stabilization of Pendulum-like Systems: A Pseudo H_∞ Control Approach

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Abstract—This paper studies the Lagrange stabilization of a class of nonlinear systems whose linear part has a singular system matrix and which have multiple periodic (in state) nonlinearities. Both state and output feedback Lagrange stabilization problems are considered. The paper develops a pseudo H_∞ control theory to solve these stabilization problems. In a similar fashion to the Strict Bounded Real Lemma in classic H_∞ control theory, a Pseudo Strict Bounded Real Lemma is established for systems with a single unstable pole. Sufficient conditions for the synthesis of state feedback and output feedback controllers are given to ensure that the closed-loop system is pseudo strict bounded real. The pseudo- H_∞ control approach is applied to solve state feedback and output feedback Lagrange stabilization problems for nonlinear systems with multiple nonlinearities. An example is given to illustrate the proposed method.

Index Terms—Pseudo- H_∞ control, Pseudo Strict Bounded Real Lemma, Pendulum-like systems, Lagrange stability.

I. INTRODUCTION

The class of pendulum-like systems is a class of nonlinear systems with periodic (in state) nonlinearities and an infinite number of equilibria [1]. They cover an important class of nonlinear systems arising in electronics, mechanics and power systems. These systems can be used to model interconnected oscillators, synchronous electrical machines and electronic phase-locked loop devices [2], [3]. An important control objective in relation to controlling such systems is to ensure that the closed-loop system retains the properties of a pendulum-like system and its trajectories are bounded, at least, in the sense of Lagrange stability. In combination with other analytical tools, this enables global asymptotic properties of the system to be established. For example, the monograph [1] makes extensive use of this approach to study global asymptotic behavior of nonlinear systems with periodic nonlinearities and an infinite number of equilibria.

The concept of Lagrange stability can be traced back to H. Poincaré's work in the 1890s [4]. In [5], Lagrange stability is defined as a property of a state x_0 of a dynamical system $\dot{x} = f(t, x)$ given on a metric space \mathcal{S} , which requires that the system trajectory $x = x(f, t, x_0)$ originating at this state x_0 to be contained in a bounded set. It is shown in [1] that if a pendulum-like system possesses both Lagrange stability and dichotomy, then it has a so-called gradient-like property. The gradient-like property guarantees that any trajectory of the

pendulum-like system eventually converges to an equilibrium. This is analogous to the asymptotic stability of a system with a single equilibrium. This observation highlights the importance of Lagrange stability as a tool to establish the gradient-like property of pendulum-like systems. It also motivates the study of pendulum-like systems within the framework of Lagrange stability which is considered in this paper.

In the authors' previous work [6], the state feedback controller synthesis problem is considered for a restricted class of pendulum-like systems in which the way that the controlled outputs enter into the nonlinearities must have a special structure. In contrast to the results in [6], this paper mainly focuses on solving the output feedback Lagrange stabilization problem for pendulum-like systems with nonlinearities which have a general structure. Unlike the special case in [6], in this more general case, a significantly different method utilizing sign-indefinite solutions to game-type Riccati equations is necessary. This has led us to develop a pseudo- H_∞ control theory to address the Lagrange stabilization problem of pendulum-like systems. This pseudo- H_∞ control theory allows a pole of the closed-loop transfer function to be located in the right half of the complex plane and ensures that the closed-loop transfer function satisfies a frequency domain condition which is similar to the bounded real property [7]. An important contribution of this paper is the pseudo strict bounded real results in Theorems 3.1 and 3.2, which are analogous to the standard strict bounded real lemma [8]. Our pseudo- H_∞ control theory can be regarded as a theory which is analogous to the standard H_∞ control theory (see [9], [10]) but with a non-standard closed-loop stability condition. Furthermore, the paper applies the proposed pseudo- H_∞ theory to solve the Lagrange stabilization problem for pendulum-like systems.

The usefulness of the Lagrange stability property of pendulum-like systems motivates research on Lagrange stabilization of pendulum-like systems; e.g., see [3], [11]–[13]. However, in these papers it was assumed that the nonlinear system contains a single nonlinearity only and has a special matched structure on its nonlinearity. This special matched structure enables the Lagrange stabilization problem to be cast as a standard H_∞ problem. In order to consider general system structures which do not satisfy matching conditions, a different approach is required which motivates our pseudo H_∞ control problem. Also, the results of [3], [11]–[13] are established using a Lagrange stability criterion given in [1] which requires the linear part of the system to be minimal. This means that a post-check is required on the linear part of the resulting closed-loop system to determine if it is minimal. In contrast, this paper uses a Lagrange stability criterion which does not have the

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minimal realization requirement but uses a strict frequency-domain condition. This Lagrange stability theory enables this paper to consider a Lagrange stabilization problem without the requirement of a post-check on the minimality of the linear part of the closed-loop system. Also, this Lagrange stability criterion allows us to solve the Lagrange stabilization problem for nonlinear systems with multiple nonlinearities. Indeed, a condition of the stability analysis techniques used in the paper is that the closed-loop system matrix A has a single zero eigenvalue, even though multiple nonlinearities are allowed. The corresponding condition on the open-loop system in our control synthesis results is that this system must have a single unobservable (or uncontrollable) mode at the origin.

To illustrate the efficacy of the proposed method, we give an example. It is concerned with Lagrange stabilization of a network of three interconnected nonlinear pendulums. Also, this system has some of the features of many practical systems such as power systems, large-scale interconnected networks and hence it suggests some application areas for the theory developed in this paper. These features are an interconnection of nonlinear but not identical elements, and the existence of multiple equilibria points due to the periodicity of the nonlinear elements.

This paper is organized as follows: Section II formulates the Lagrange stabilization problem for pendulum-like systems; Section III presents a pseudo H_∞ control theory, which is motivated by the problem formulated in Section 2; Section IV presents our main results on output feedback Lagrange stabilization of unobservable pendulum-like systems; Section V presents our results on the output feedback Lagrange stabilization of uncontrollable pendulum-like systems; Section VI gives results on the state feedback Lagrange stabilization of uncontrollable pendulum-like systems. Section VII presents an example to illustrate the efficacy of the proposed method and Section VIII concludes this paper. All of the proofs of the theorems in the Sections II-VI are contained in the Appendix.

Notation: \mathcal{Z} denotes the set of integers. $\mathcal{R}^{n \times m}$ and $\mathcal{C}^{n \times m}$ denote the space of $n \times m$ real matrices and the space of $n \times m$ complex matrices, respectively. \mathcal{Q} denotes the set of rational numbers and \mathcal{Q}^m denotes the set of vectors of m rational numbers. $\sigma(A)$ denotes the set of the eigenvalues of a matrix A . $\sigma_{\max}[\cdot]$ denotes the maximum singular value of a matrix. \mathcal{RH}_∞ denotes the space of all proper and real rational stable transfer function matrices. \mathcal{R}_+ denotes the set of positive real numbers and $\mathcal{R}_+^n = (\mathcal{R}_+)^n$. $\rho(X)$ denotes the spectral radius of the matrix X . $\text{diag}[a_1, \dots, a_n]$ is a diagonal matrix with a_1, \dots, a_n as its diagonal elements. $\mathbb{B}(a, \varepsilon)$ denotes a neighborhood around $a \in \mathcal{R}^n$, defined as $\{\tilde{a} \in \mathcal{R}^n : \|\tilde{a} - a\| < \varepsilon\}$. Given a vector $\tau = [\tau_1, \dots, \tau_m]^T \in \mathcal{R}_+^m$, M_τ denotes the diagonal matrix $M_\tau = \text{diag}[\tau_1, \dots, \tau_m]$. Similarly, $M_\mu = \text{diag}[\mu_1, \dots, \mu_m]$. Given a vector $v \in \mathcal{Q}^m$, $\text{LCMD}(v)$ denotes the least common multiple (LCM) of the denominators of all the elements of v .

II. PROBLEM FORMULATION OF LAGRANGE STABILIZATION FOR PENDULUM-LIKE SYSTEM

A. Pendulum-like Systems

We consider a class of nonlinear systems defined as follows:

$$\begin{aligned}\dot{x} &= Ax + Bw, \\ z &= Cx,\end{aligned}\tag{1}$$

where $x \in \mathcal{R}^n$ is the state, $z \in \mathcal{R}^m$ is the nonlinearity output vector and $w \in \mathcal{R}^m$ is the nonlinearity input vector. Also, $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times m}$, $C = [C_1^T, \dots, C_m^T]^T \in \mathcal{R}^{m \times n}$, $C_i \in \mathcal{R}^{1 \times n}$, $i = 1, \dots, m$. The components of the vector $w = [w_1, \dots, w_m]^T$ are determined from the corresponding components of the vector $z = [z_1, \dots, z_m]^T$ via nonlinear functions

$$w_i = \phi_i(t, z_i)\tag{2}$$

where $\phi_i : \mathcal{R}_+ \times \mathcal{R} \rightarrow \mathcal{R}$ is a continuous, locally Lipschitz in the second argument and periodic function with period $\Delta_i > 0$; i.e.,

$$\phi_i(t, z_i + \Delta_i) = \phi_i(t, z_i), \quad \forall t \in \mathcal{R}_+, z_i \in \mathcal{R}.\tag{3}$$

This type of nonlinearity appears frequently in the practical engineering systems mentioned in Section I. Phase-locked loops [14] and a pendulum system with a vibrating point of suspension [1] are typical examples of such systems. We also refer to the example given in Section VII. The transfer function of the linear part of the system (1) is given by $G(s) = C(sI - A)^{-1}B$. The nonlinear functions $\phi_i(t, z_i)$, $i = 1, \dots, m$, are assumed to satisfy the sector conditions,

$$-\mu_i \leq \frac{\phi(t, z_i)}{z_i} \leq \mu_i, \quad \forall t \in \mathcal{R}_+, z_i \neq 0,\tag{4}$$

where $\mu_i \in \mathcal{R}_+$, $i = 1, \dots, m$.

We define $\Delta \in \mathcal{R}^{m \times m}$ as $\Delta = \text{diag}[\Delta_1, \dots, \Delta_m]$. Given a vector $d \in \mathcal{R}^n$, let $\Pi(d) \triangleq \{kd | k \in \mathcal{Z}\}$.

Definition 2.1: (Pendulum-like System [1]) The nonlinear system (1), (2), (3) is pendulum-like with respect to $\Pi(d)$ if for any solution $x(t, t_0, x_0)$ of (1), (2), (3) with $x(t_0) = x_0$, we have $x(t, t_0, x_0) + \tilde{d} = x(t, t_0, x_0 + \tilde{d})$, for all $t \geq t_0$, and all $\tilde{d} \in \Pi(d)$.

Remark 2.1: This definition reflects the fact that the phase portrait of a pendulum-like system is periodic. For example, in the case of a simple pendulum, this means that its position variable can be represented by an angle between 0 and 2π .

Definition 2.2: (Lagrange Stability [1]) The nonlinear system (1), (2) is said to be Lagrange stable if all its solutions are bounded.

B. Lagrange Stabilization Problem for Pendulum-like Systems

The pendulum-like system to be stabilized will be a controlled version of the nonlinear system (1), (2), (3), (4). That is, the linear part of the system is described by the state equations

$$\dot{x} = Ax + B_2u + B_1w,\tag{5a}$$

$$z = C_1x + D_{12}u,\tag{5b}$$

$$y = C_2x + D_{21}w,\tag{5c}$$

where $x \in \mathcal{R}^n$, $w \in \mathcal{R}^m$, $z \in \mathcal{R}^m$ are defined as in (1), $u \in \mathcal{R}^q$ is the control input, and $y \in \mathcal{R}^p$ is the measured output. Here, all the matrices are assumed to have compatible dimensions. Also, the components of the nonlinearity input w are related to the components of the system output z as in (2) and the nonlinearities ϕ_i have the property (3). Furthermore, the nonlinearities are assumed to satisfy the sector condition (4). The system block diagram is shown in Figure 1.

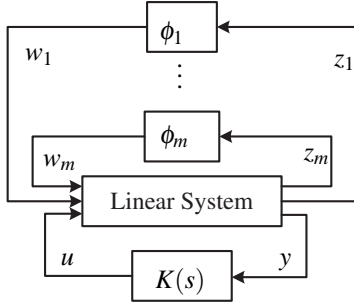


Fig. 1. Nonlinear control system with periodic nonlinearities.

Problem 1: (Output Feedback Lagrange Stabilization) The output feedback Lagrange stabilization problem for the nonlinear system (5), (2), (3), (4) is to design a linear controller with the transfer function $K(s)$ and state-space realization:

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c \end{aligned} \quad (6)$$

such that the resulting closed-loop system is pendulum-like and Lagrange stable.

Problem 2: (State Feedback Lagrange Stabilization) The state feedback Lagrange stabilization problem is to design a state feedback control law $u = Kx$ for the system (5a), (5b), (2), (3), (4) to ensure that the resulting closed-loop system is pendulum-like and Lagrange stable.

Note that in some cases, it may be possible to design a controller in the form of (6) to asymptotically stabilize the system (5), (2), (4). Such cases are trivial from the point of view of Lagrange stabilization. In order to rule out these trivial cases and to guarantee that the closed-loop system is a pendulum-like system, we will assume that the linear part of the systems (5) has uncontrollable or unobservable modes.

To solve the above two problems, the following two technical results of [6] will be used:

Lemma 2.1: ([6]) Consider the nonlinear system (1), (2), (3). Suppose $\det A = 0$ and there exists a vector $\bar{d} \neq 0$ such that $A\bar{d} = 0$, $C_i \bar{d} \neq 0$, $i = 1, \dots, m$, and $(\Delta)^{-1} C \bar{d} \in \mathcal{Q}^m$. Also, let $\frac{\Delta_i}{C_i \bar{d}} = \frac{p_i}{q_i}$ for all $i = 1, \dots, m$, where $p_i, q_i \neq 0$ are integers. Let \bar{p} be the LCM of p_i , $i = 1, \dots, m$. Then, the system (1), (2), (3) is pendulum-like with respect to $\Pi(\bar{d})$ where $d = \bar{p}\bar{d}$.

Lemma 2.2: ([6]) (*Lagrange Stability Criterion*) Suppose the system (1), (2), (3), (4) is a pendulum-like system. Also, suppose there exist a constant $\lambda > 0$ and a vector $\tau = [\tau_1, \dots, \tau_m]^T \in \mathcal{R}_+^m$ satisfying the following conditions:

- i. $A + \lambda I$ has $n - 1$ eigenvalues with negative real parts and one with positive real part;

- ii. $G^T(-j\omega - \lambda)M_\tau G(j\omega - \lambda) < M_\mu^{-1}M_\tau M_\mu^{-1}$, for all $\omega \geq 0$.

Then, the nonlinear system (1), (2), (3), (4) is Lagrange stable.

The proofs of these two results appear in the journal version of [6] but are included in the Appendix for completeness.

Lemma 2.2 is the key result to establish Lagrange stability of the closed-loop systems under consideration. It involves a frequency domain condition, which is similar to the bounded real property in [7], and a system state matrix $A + \lambda I$ which has one unstable eigenvalue. However, it does not require the minimality of the linear part of the system (1). To establish these conditions in the Lagrange stabilization problems 1 and 2, we develop a pseudo- H_∞ control theory in the next section, which is analogous to the standard H_∞ control theory.

III. PSEUDO- H_∞ CONTROL

A. The Pseudo Strict Bounded Real Property and the Corresponding Strict Bounded Real Lemma (SBRL)

The bounded real property is an important concept frequently used in the standard H_∞ control theory. We begin our development of pseudo H_∞ control with the definition of the pseudo strict bounded real property, which is analogous to the standard bounded real property.

Definition 3.1: A matrix $A \in \mathcal{R}^{n \times n}$ which has $n - 1$ eigenvalues with negative real parts and one eigenvalue with positive real part is said to be *pseudo-Hurwitz*. A symmetric matrix $P \in \mathcal{R}^{n \times n}$ is said to be *pseudo-positive definite* if it has $n - 1$ positive eigenvalues and one negative eigenvalue.

Definition 3.2: A linear time-invariant (LTI) system (1) is called *pseudo strict bounded real* if the following conditions hold:

- (i) A is pseudo Hurwitz;
- (ii)

$$\max_{\omega \in \mathcal{R}} \{ \sigma_{\max}[G(-j\omega)^T G(j\omega)] \} < 1. \quad (7)$$

Theorem 3.1: Consider the LTI system (1). If the Riccati equation

$$A^T P + PA + PBB^T P + C^T C = 0 \quad (8)$$

has a solution $P = P^T$ such that P is pseudo-positive definite and $A + BB^T P$ has no purely imaginary eigenvalues, then the system (1) is pseudo strict bounded real.

Theorem 3.2: If the LTI system (1) is pseudo strict bounded real, then

- 1) There exists a pseudo-positive definite matrix $P = P^T$ such that

$$A^T P + PA + PBB^T P + C^T C < 0. \quad (9)$$

- 2) Furthermore, if in addition the pair (A, B) is stabilizable and the pair (A, C) is observable, then the Riccati equation (8) has a stabilizing solution P which is pseudo-positive definite.

Theorem 3.1 is analogous to the sufficiency part of the strict bounded real lemma for systems with non-minimal realizations [8]. Also, Theorem 3.2 is analogous to the necessity part of the strict bounded real lemma for systems with non-minimal

realizations. Theorems 3.1 and 3.2 are together called the *pseudo strict bounded real lemma*.

The pseudo strict bounded real lemma gives a relationship between state-space conditions, such as solvability of (8) and pseudo-Hurwitzness of A , and the frequency-domain inequality (7). This will allow us to replace the frequency domain condition for the closed-loop system that will appear in the application of Lemma 2.2, with a condition in the state-space form. This is a key step in the derivation of a solution to Problems 1 and 2.

B. State Feedback Pseudo- H_∞ Control

The state feedback pseudo- H_∞ control problem for the LTI system (5a), (5b) involves designing a state feedback law $u = Kx$ which ensures that the corresponding closed-loop system is pseudo strict bounded real. In an analogous way to H_∞ control theory [9], [10], the main result of this section presented in the following theorem, gives a sufficient condition for the existence of a solution to the problem.

The following assumption is made on the system (5a), (5b):

Assumption 3.1: $E_1 = D_{12}^T D_{12} > 0$.

Theorem 3.3: Suppose Assumption 3.1 holds for the system (5a), (5b) and the Riccati equation

$$(A - B_2 E_1^{-1} D_{12}^T C_1)^T P + P(A - B_2 E_1^{-1} D_{12}^T C_1) + P(B_1 B_1^T - B_2 E_1^{-1} B_2^T)P + C_1^T (I - D_{12} E_1^{-1} D_{12}^T) C_1 = 0 \quad (10)$$

has a solution $P = P^T$ such that P is pseudo-positive definite and the matrix

$$A - B_2 E_1^{-1} D_{12}^T C_1 + (B_1 B_1^T - B_2 E_1^{-1} B_2^T)P \quad (11)$$

has no purely imaginary eigenvalues. Then, the state feedback control law

$$u = -E_1^{-1} (B_2^T P + D_{12}^T C_1) x \quad (12)$$

solves the state feedback pseudo- H_∞ control problem. That is, the resulting closed-loop system is pseudo strict bounded real.

Remark 3.1: In practice, it is usually convenient to use the stabilizing solution to the Riccati equation (10) in order to construct the required state feedback control law (12).

C. Output Feedback Pseudo- H_∞ Control

Analogous to the standard output feedback H_∞ control problem, the output feedback pseudo- H_∞ control problem for the system (5) involves designing a compensator of the form (6) to make the corresponding closed-loop system pseudo strict bounded real. The following two theorems each give a sufficient condition for the existence of a solution to the output feedback pseudo- H_∞ control problem for a system of the form (5). Besides Assumption 3.1, the following assumption is also made on the system (5):

Assumption 3.2: $E_2 = D_{21} D_{21}^T > 0$.

Theorem 3.4: Suppose the system (5) satisfies Assumptions 3.1 and 3.2 and the following conditions are satisfied:

- (i) The Riccati equation

$$(A - B_2 E_1^{-1} D_{12}^T C_1)^T X + X(A - B_2 E_1^{-1} D_{12}^T C_1) + X(B_1 B_1^T - B_2 E_1^{-1} B_2^T)X + C_1^T (I - D_{12} E_1^{-1} D_{12}^T) C_1 = 0 \quad (13)$$

has a stabilizing solution $X = X^T$ which is pseudo-positive definite;

- (ii) The Riccati equation

$$(A - B_1 D_{21}^T E_2^{-1} C_2)Y + Y(A - B_1 D_{21}^T E_2^{-1} C_2)^T + Y(C_1^T C_1 - C_2^T E_2^{-1} C_2)Y + B_1(I - D_{21}^T E_2^{-1} D_{21})B_1^T = 0 \quad (14)$$

has a stabilizing solution $Y = Y^T$ which is positive definite;

- (iii) The matrix XY has a spectral radius strictly less than one, $\rho(XY) < 1$.

Then, there exists a dynamic output feedback compensator of the form (6) such that the resulting closed-loop system is pseudo strict bounded real. Furthermore, the matrices defining the required dynamic feedback controller (6) can be constructed as follows:

$$\begin{aligned} A_c &= A + B_2 C_c - B_c C_2 + (B_1 - B_c D_{21})B_1^T X, \\ B_c &= (I - YX)^{-1}(YC_2^T + B_1 D_{21}^T)E_2^{-1}, \\ C_c &= -E_1^{-1}(B_2^T X + D_{12}^T C_1). \end{aligned} \quad (15)$$

Theorem 3.5: Suppose the system (5) satisfies Assumptions 3.1 and 3.2 and the following conditions are satisfied:

- (i) The Riccati equation (13) has a positive definite stabilizing solution $X = X^T$;
- (ii) the Riccati equation (14) has a pseudo-positive definite stabilizing solution $Y = Y^T$;
- (iii) The matrix XY has a spectral radius strictly less than one, $\rho(XY) < 1$.

Then, there exists a dynamic output feedback compensator of the form (6) such that the resulting closed-loop system is pseudo strict bounded real. Furthermore, the matrices in the required dynamic feedback controller (6) can be constructed as follows:

$$\begin{aligned} A_c &= A + B_c C_2 - B_2 C_c + Y C_1^T (C_1 - D_{12} C_c), \\ B_c &= -(Y C_2^T + B_1^T D_{21}^T)E_2^{-1}, \\ C_c &= E_1^{-1}(B_2^T X + D_{12}^T C_1)(I - YX)^{-1}. \end{aligned} \quad (16)$$

Remark 3.2: According to [15], the stabilizing solutions to the Riccati equations (13) and (14) are unique, if they exist.

IV. OUTPUT FEEDBACK LAGRANGE STABILIZING CONTROLLER SYNTHESIS FOR UNOBSERVABLE SYSTEMS

In this section, the output feedback pseudo H_∞ control theory developed in the previous section is used to solve Problem 1 for nonlinear systems satisfying the following assumptions, which will be used to ensure that the closed-loop system is pendulum-like and to rule out trivial cases in which the nonlinear system can be asymptotically stabilized:

Assumption 4.1: There exists a non-zero vector x such that $Ax = 0$ and $C_2 x = 0$.

Assumption 4.1 implies that (A, C_2) is unobservable and the origin is an unobservable mode. Using the Kalman decomposition in the unobservable form [16], it follows that there exists a non-singular state-space transformation matrix T such that the system matrices of the system (5) are transformed to the form

$$\begin{aligned}\tilde{A} &= T^{-1}AT = \begin{bmatrix} \tilde{A}_1 & 0 \\ \tilde{A}_2 & 0 \end{bmatrix}, \quad \tilde{B}_2 = T^{-1}B_2 = \begin{bmatrix} \tilde{B}_{2a} \\ \tilde{B}_{2b} \end{bmatrix}, \\ \tilde{B}_1 &= T^{-1}B_1 = \begin{bmatrix} \tilde{B}_{1a} \\ \tilde{B}_{1b} \end{bmatrix}, \\ \tilde{C}_1 &= C_1T = [\tilde{C}_{1a} \quad \tilde{C}_{1b}], \quad \tilde{D}_{12} = D_{12}, \\ \tilde{C}_2 &= C_2T = [\tilde{C}_{2a} \quad 0], \quad \tilde{D}_{21} = D_{21},\end{aligned}\quad (17)$$

where $\tilde{A}_1 \in \mathcal{R}^{(n-l) \times (n-l)}$, $\tilde{B}_{1a} \in \mathcal{R}^{(n-l) \times m}$, $\tilde{B}_{2a} \in \mathcal{R}^{(n-l) \times q}$, $\tilde{C}_{1a}, \tilde{C}_{2a} \in \mathcal{R}^{m \times (n-l)}$.

Also, let $e_n = [0_{1 \times (n-1)} \quad 1]^T$. We define two vectors $\chi = C_1Te_n$ and $\bar{d} = [0_{1 \times n} \quad e_n^T T^T]^T \in \mathcal{R}^{2n}$.

Assumption 4.2: There exists a constant $\tau_0 > 0$ such that all the elements of the vector $v = \tau_0 \Delta^{-1} \chi$ are non-zero rational numbers.

Remark 4.1: In the case where the coefficients in the system (5) are all rational numbers, Assumption 4.2 amounts to an assumption that the periods of the nonlinearities are commensurate.

The main result of this section involves the following Riccati equations dependent on parameters $\lambda > 0$ and $\tau_i > 0$, $i = 1, \dots, m$:

$$\begin{aligned}(\lambda I + A - B_2 \bar{E}_1^{-1} D_{12}^T M_\tau C_1)^T X + X(\lambda I + A - B_2 \bar{E}_1^{-1} D_{12}^T M_\tau C_1) \\ + X(B_1 M_\mu M_\tau^{-1} M_\mu B_1^T - B_2 \bar{E}_1^{-1} B_2^T) X \\ + C_1^T (M_\tau - M_\tau D_{12} \bar{E}_1^{-1} D_{12}^T M_\tau) C_1 = 0,\end{aligned}\quad (18)$$

$$\begin{aligned}(\lambda I + A - B_1 M_\mu M_\tau^{-1} M_\mu D_{21}^T \bar{E}_2^{-1} C_2) Y \\ + Y(\lambda I + A - B_1 M_\mu M_\tau^{-1} M_\mu D_{21}^T \bar{E}_2^{-1} C_2)^T \\ + B_1 \left(\begin{array}{c} M_\mu M_\tau^{-1} M_\mu B_1^T \\ -M_\mu M_\tau^{-1} M_\mu D_{21}^T \bar{E}_2^{-1} D_{21} M_\mu M_\tau^{-1} M_\mu \end{array} \right) B_1^T \\ + Y(C_1^T M_\tau C_1 - C_2^T \bar{E}_2^{-1} C_2) Y = 0,\end{aligned}\quad (19)$$

where $\bar{E}_1 = D_{12}^T M_\tau D_{12}$ and $\bar{E}_2 = D_{21} M_\mu M_\tau^{-1} M_\mu D_{21}^T$. If these Riccati equations have suitable solutions, we will define the parameter matrices of the controller (6) as follows:

$$\begin{aligned}A_c &= A + B_c C_2 - B_2 C_c + Y C_1^T (M_\tau C_1 - M_\tau D_{12} C_c), \\ B_c &= -(Y C_2^T + B_1 M_\mu M_\tau^{-1} M_\mu D_{21}^T) \bar{E}_2^{-1}, \\ C_c &= \bar{E}_1^{-1} (B_2^T X + D_{12}^T M_\tau C_1) (I - YX)^{-1}.\end{aligned}\quad (20)$$

The following theorem, which is the main result of this paper, gives a sufficient condition for the existence of a Lagrange stabilizing controller for the nonlinear system (5), (2), (3), (4) :

Theorem 4.1: Suppose Assumptions 3.1, 3.2, 4.1 and 4.2 hold for the nonlinear system (5), (2), (3), (4). Also, suppose there exist constants $\lambda > 0$ and $\tau_i > 0$, $i = 0, \dots, m$ such that the following conditions are satisfied:

- I. The Riccati equation (18) has a stabilizing solution $X = X^T$ which is positive definite;

- II. The Riccati equation (19) has a pseudo-positive definite stabilizing solution $Y = Y^T$;
- III. The matrix XY has a spectral radius strictly less than one, $\rho(XY) < 1$.

Then, the resulting closed-loop system corresponding to the controller (6), (20) is a pendulum-like system with respect to $\Pi(\tau_0 \bar{p} \bar{d})$ and is Lagrange stable. Here $\bar{p} = \text{LCMD}(v)$.

V. OUTPUT FEEDBACK LAGRANGE STABILIZING CONTROLLER SYNTHESIS FOR UNCONTROLLABLE SYSTEMS

In this section, the state feedback and output feedback pseudo H_∞ control theories in Section III are applied to Lagrange stabilization for nonlinear systems satisfying the following assumption which is dual to Assumption 4.1:

Assumption 5.1: There exists a non-zero vector x such that $x^T A = 0$ and $x^T B_2 = 0$.

In a similar way to Assumption 4.1, this assumption is also used to ensure that the closed-loop system is pendulum-like and to rule out trivial cases in which the system can be asymptotically stabilized. Also, this assumption implies that (A, B_2) is not controllable. Using the Kalman Decomposition [16], it follows from Assumption 5.1 that there exists a non-singular state-space transformation matrix \bar{T} such that the matrices of the system (5) are transformed to the form

$$\begin{aligned}\tilde{A} &= \bar{T}^{-1}AT = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_2 = \bar{T}^{-1}B_2 = \begin{bmatrix} \tilde{B}_{2a} \\ 0 \end{bmatrix}, \\ \tilde{B}_1 &= \bar{T}^{-1}B_1 = \begin{bmatrix} \tilde{B}_{1a} \\ \tilde{B}_{1b} \end{bmatrix}, \\ \tilde{D}_{12} &= D_{12}, \quad \tilde{C}_1 = C_1 \bar{T} = [\tilde{C}_{1a} \quad \tilde{C}_{1b}], \\ \tilde{C}_2 &= C_2 \bar{T} = [\tilde{C}_{2a} \quad \tilde{C}_{2b}], \quad \tilde{D}_{21} = D_{21},\end{aligned}\quad (21)$$

where $\tilde{A}_1 \in \mathcal{R}^{(n-l) \times (n-l)}$, $\tilde{A}_2 \in \mathcal{R}^{(n-l) \times l}$, $\tilde{B}_{2a} \in \mathcal{R}^{(n-l) \times q}$, $\tilde{B}_{1a} \in \mathcal{R}^{(n-l) \times m}$, $\tilde{C}_{1a}, \tilde{C}_{2a} \in \mathcal{R}^{m \times (n-l)}$.

A. Output Feedback Lagrange Stabilization for Uncontrollable Systems

The main result of this section involves the Riccati equations (18) and (19) which are dependent on parameters $\lambda > 0$ and $\tau_i > 0$, $i = 1, \dots, m$. Using solutions X and Y to the equations (18) and (19), we can construct the following matrices:

$$\begin{aligned}A_c &= A + B_2 C_c - B_c C_2 \\ &\quad + (B_1 M_\mu M_\tau^{-1} M_\mu - B_c D_{21} M_\mu M_\tau^{-1} M_\mu) B_1^T X, \\ B_c &= (I - YX)^{-1} (Y C_2^T + B_1 M_\mu M_\tau^{-1} M_\mu D_{21}^T) \bar{E}_2^{-1}, \\ C_c &= -\bar{E}_1^{-1} (B_2^T X + D_{12}^T M_\tau C_1).\end{aligned}\quad (22)$$

Also, we define two vectors of constants:

$$\begin{aligned}\bar{d}_0 &= - \begin{bmatrix} A_c & B_c \tilde{C}_{2a} \\ \tilde{B}_{2a} C_c & \tilde{A}_1 \end{bmatrix}^{-1} \begin{bmatrix} B_c \tilde{C}_{2b} \\ \tilde{A}_2 \end{bmatrix}, \\ \chi &= [\quad -D_{12} \bar{E}_1^{-1} (B_2^T X + D_{12}^T M_\tau C_1) \quad C_1 \quad] \bar{d},\end{aligned}\quad (23)$$

where $\bar{d} = \begin{bmatrix} I & 0 \\ 0 & \bar{T} \end{bmatrix} \begin{bmatrix} \bar{d}_0 \\ 1 \end{bmatrix}$ with \bar{T} defined in the Kalman decomposition (21). Using this notation, a sufficient condition

for the solution to the output feedback Lagrange stabilization Problem 1 can now be presented:

Theorem 5.1: Suppose Assumptions 3.1, 3.2 and 5.1 hold for the system (5), (2), (3), (4). Also, suppose there exist constants $\lambda > 0$ and τ_i , $i = 0, \dots, m$ such that the following conditions are satisfied for the nonlinear system (5), (2), (3), (4):

- I. The Riccati equation (18) has a stabilizing pseudo-positive definite solution $X = X^T$;
- II. The Riccati equation (19) has a stabilizing solution $Y = Y^T$ which is positive definite;
- III. The matrix XY has a spectral radius strictly less than one, $\rho(XY) < 1$;
- IV. The matrix $\begin{bmatrix} A_c & B_c \tilde{C}_{2a} \\ \tilde{B}_{2a} C_c & \tilde{A}_1 \end{bmatrix}$ is non-singular and all the elements of the vector $v = \tau_0 \Delta^{-1} \chi$ are non-zero rational numbers, where A_c , B_c , C_c and χ are defined in (22) and (23) using X , Y in I, II and III.

Then, the closed-loop system consisting of the system (5), (2), (3), (4) and the controller (6), (22) is a pendulum-like system with respect to $\Pi_\lambda = \{\bar{p}\tau_0\bar{d}\}$ and is Lagrange stable. Here $\bar{p} = \text{LCMD}(v)$.

B. Satisfaction of the rationality condition.

Theorem 5.1 gives sufficient conditions for the existence of a solution to the Lagrange stabilizing controller synthesis problem for a nonlinear system satisfying Assumption 5.1. However, the question arises as to whether, given $\lambda > 0$, there will exist positive constants τ_i , $0 = 1, \dots, m$, such that the stabilizing solutions to the Riccati equations (18) and (19) satisfy the rationality condition IV of this theorem.

First, we demonstrate that such $\tau = [\tau_1, \dots, \tau_m]^T$, if exists, can be constrained to be a unit vector. Given any $\gamma > 0$, let $\hat{\tau} = \gamma\tau$, $\tilde{X} = \gamma X$, $\tilde{Y} = \gamma^{-1}Y$, $\tilde{M}_\tau = \gamma M_\tau$, $\tilde{E}_1 = D_{12}^T M_\tau D_{12}$ and $\tilde{E}_2 = D_{21} M_\tau^{-1} D_{21}^T$. Multiplying the Riccati equation (18) by γ and multiplying (19) by γ^{-1} gives that

$$\begin{aligned} & (A + \lambda I - B_2 \tilde{E}_1^{-1} D_{12}^T M_\tau C_1)^T \tilde{X} + \tilde{X} (A + \lambda I - B_2 \tilde{E}_1^{-1} D_{12}^T M_\tau C_1) \\ & + C_1^T (\tilde{M}_\tau - \tilde{M}_\tau D_{12} \tilde{E}_1^{-1} D_{12}^T M_\tau) C_1 \\ & + \tilde{X} (B_1 M_\mu M_\tau^{-1} M_\mu B_1^T - B_2 \tilde{E}_1^{-1} B_2^T) \tilde{X} = 0, \\ & (\lambda I + A - B_1 M_\mu M_\tau^{-1} M_\mu D_{21}^T \tilde{E}_2^{-1} C_2) \tilde{Y} \\ & + \tilde{Y} (\lambda I + A - B_1 M_\mu M_\tau^{-1} M_\mu D_{21}^T \tilde{E}_2^{-1} C_2)^T \\ & + \tilde{Y} (C_1^T \tilde{M}_\tau C_1 - C_2^T \tilde{E}_2^{-1} C_2) \tilde{Y} \\ & + B_1 (M_\mu M_\tau^{-1} M_\mu - M_\mu M_\tau^{-1} D_{21}^T \tilde{E}_2^{-1} D_{21} M_\tau^{-1} M_\mu) B_1^T = 0. \end{aligned} \quad (24)$$

It is obvious that (24) has the same form as (18) but both X and M_τ are scaled by γ . Also, (25) has the same form as (19) but Y is scaled by γ^{-1} and M_τ is scaled by γ . Hence, Conditions I-III in the statement of Theorem 5.1 are not affected if we use \tilde{X} , \tilde{Y} , \tilde{M}_τ , \tilde{E}_1 and \tilde{E}_2 to replace X , Y , M_τ , \tilde{E}_1 and \tilde{E}_2 respectively. In addition, it is straightforward to verify that Condition IV of Theorem 5.1 is not affected by scaling the vector of constants τ . Thus, without loss of generality, we assume that τ is a unit vector throughout the remainder of this section, and if we take

$\tau_i > 0$, $i = 1, \dots, m-1$ as independent constants combined into the vector $\tilde{\tau} = [\tau_1, \dots, \tau_{m-1}]$, then τ_m is given by

$$\tau_m = \sqrt{1 - \sum_{i=1}^{m-1} \tau_i^2}. \quad (26)$$

Define

$$\mathbb{T} = \left\{ \tilde{\tau} \in \mathcal{R}^{m-1} : \begin{array}{l} \text{Equations (18) and (19) have} \\ \text{nonsingular stabilizing solutions} \end{array} \right\}.$$

Let $\tilde{\tau} = [\tau_0, \dots, \tau_{m-1}]^T = [\tau_0, \tilde{\tau}^T]^T$ and define a function $f(\tilde{\tau}) = [f_1(\tilde{\tau}) \dots f_m(\tilde{\tau})]^T = \tau_0 \Delta^{-1} \chi$ on the set $\mathbb{F} = \{\tilde{\tau} : \tau_0 > 0, \tilde{\tau} \in \mathbb{T}\}$. Let $J(\tau_0, \tau_1, \dots, \tau_{m-1})$ be the Jacobian matrix of $f(\tilde{\tau})$,

$$J(\tau_0, \tau_1, \dots, \tau_{m-1}) = \begin{bmatrix} \frac{\partial f_1}{\partial \tau_0} & \frac{\partial f_1}{\partial \tau_1} & \dots & \frac{\partial f_1}{\partial \tau_{m-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial \tau_0} & \frac{\partial f_m}{\partial \tau_1} & \dots & \frac{\partial f_m}{\partial \tau_{m-1}} \end{bmatrix}. \quad (27)$$

Then, we have $J(\tilde{\tau}) = \Delta^{-1} \tilde{J}(\tilde{\tau})$ and the elements of $\tilde{J}(\tilde{\tau})$ are

$$\begin{aligned} \tilde{J}_{i,1} &= w_i, \quad i = 1, \dots, m; \\ \tilde{J}_{i,j} &= \begin{cases} \tau_0 \left(\tau_i^{-2} w_i + \tau_i^{-1} \frac{\partial w_i}{\partial \tau_i} \right) : i = j, i = 1, \dots, m-1; \\ \tau_0 \tau_i^{-1} \frac{\partial w_i}{\partial \tau_j} : i, j = 1, \dots, m-1, i \neq j; \end{cases} \\ \tilde{J}_{m,j} &= \tau_0 \left(\tau_m^{-3} \tau_j w_m + \tau_m^{-1} \frac{\partial w_m}{\partial \tau_j} \right), \quad j = 1, \dots, m. \end{aligned} \quad (28)$$

The following theorem gives a sufficient condition for the existence of the constants τ_0, \dots, τ_m satisfying all the conditions of Theorem 5.1:

Theorem 5.2: Suppose Assumptions 3.1, 3.2 and 5.1 hold for the system (5), (2), (3), (4). Also, suppose there exist a constant $\lambda > 0$ and a vector of positive constants $\tilde{\tau} = [\tau_0, \dots, \tau_{m-1}]$ such that the following conditions are satisfied for the system (5), (2), (3), (4):

- (I) Conditions I, II and III of Theorem 5.1 hold;
- (II) $\det \tilde{J}(\tilde{\tau}) \neq 0$ where the elements of $\tilde{J}(\tilde{\tau})$ are defined as (28).

Then, given any sufficiently small $\varepsilon > 0$, there exists $\check{\tau} = [\check{\tau}_0, \check{\tau}_1, \dots, \check{\tau}_{m-1}] \in \mathbb{F}$ such that $\|\check{\tau} - \tilde{\tau}\| < \varepsilon$ and the constants $\tau_0 = \check{\tau}_0$, $\tau_i = \check{\tau}_i$, $i = 1, \dots, m-1$ and τ_m (defined as in (26)) satisfy all the conditions of Theorem 5.1 and hence the corresponding closed-loop system is pendulum-like and Lagrange stable.

VI. STATE FEEDBACK LAGRANGE STABILIZATION FOR UNCONTROLLABLE SYSTEMS

In this section, we give a sufficient condition for the existence of a solution to the state feedback Lagrange stabilization problem (Problem 2) of Section II.

Using a solution to the Riccati equation (18), we define two vectors $\bar{d} = \bar{T}[\bar{d}_0^T \ 1]^T$ and $\chi = [\chi_1 \dots \chi_m]^T = ((I - D_{12} \tilde{E}_1^{-1} D_{12}^T M_\tau) C_1 - D_{12} \tilde{E}_1^{-1} B_2^T X) \bar{d}$, where \bar{T} is defined by (21) and

$$\begin{aligned} \bar{d}_0 &= -(\tilde{A}_1 - \tilde{B}_{2a} \tilde{E}_1^{-1} \tilde{D}_{12}^T M_\tau C_{1a} - \tilde{B}_{2a} \tilde{E}_1^{-1} \tilde{B}_{2a}^T \tilde{X}_{11})^{-1} \\ &\quad \times (\tilde{A}_2 - \tilde{B}_{2a} \tilde{E}_1^{-1} \tilde{D}_{12}^T M_\tau C_{1b} - \tilde{B}_{2a} \tilde{E}_1^{-1} \tilde{B}_{2a}^T \tilde{X}_{12}) \end{aligned}$$

with $\bar{X}_{11} \in \mathcal{R}^{(n-1) \times (m-1)}$, $\bar{X}_{12} \in \mathcal{R}^{(n-1) \times 1}$ defined by $\bar{T}^T X \bar{T} = \begin{bmatrix} \bar{X}_{11} & \bar{X}_{12} \\ \bar{X}_{12}^T & \bar{X}_{22} \end{bmatrix}$.

Theorem 6.1: Consider the nonlinear system (5a), (5b), (2), (3), (4) and suppose Assumptions 3.1 and 5.1 are satisfied. If there exist constants $\lambda > 0$ and $\tau_i > 0$, $i = 0, \dots, m$ such that the Riccati equation (18) has a pseudo-positive definite solution $X = X^T$ such that

- I. The matrix $A + \lambda I - B_2 \bar{E}_1^{-1} D_{12}^T M_\tau C_1 + (B_1 M_\mu M_\tau^{-1} M_\mu B_1^T - B_2 \bar{E}_1^{-1} B_2^T) X$ is Hurwitz;
- II. All elements of the vector $v = \tau_0 \Delta^{-1} \chi$ are non-zero rational numbers.

Then, the closed-loop system corresponding to the state feedback control

$$u = (-D_{12} \bar{E}_1^{-1} D_{12}^T M_\tau C_1 - D_{12} \bar{E}_1^{-1} B_2^T P) x \quad (29)$$

is a pendulum-like system with respect to $\Pi(\bar{p} \tau_0 \bar{d})$ and is Lagrange stable, where $\bar{p} = \text{LCMD}(v)$.

In a similar way to Theorem 5.2, a sufficient condition for the existence of constants τ_0, \dots, τ_m satisfying Condition II of Theorem 6.1 is now given. The proof of this result is similar to that of Theorem 5.2 and is omitted.

Theorem 6.2: Consider the system (5a), (5b), (2), (3), (4) and suppose Assumptions 3.1, 5.1 are satisfied. Also, suppose there exists a constant $\lambda > 0$ and a vector of positive constants $\tilde{\tau} = [\tau_0, \dots, \tau_{m-1}]^T$ satisfying the following conditions:

- I. The Riccati equation (18) has a pseudo-positive definite stabilizing solution X ;
- II. $\tilde{J}(\tilde{\tau}) \neq 0$ where $\tilde{J}(\tilde{\tau})$ is defined in (28).

Then, given any sufficiently small $\varepsilon > 0$, there exists a $\tilde{\tau} = [\tilde{\tau}_0, \tilde{\tau}_1, \dots, \tilde{\tau}_{m-1}] \in \mathbb{F}$ such that $\|\tilde{\tau} - \tilde{\tau}\| < \varepsilon$ and the constants $\tau_0 = \tilde{\tau}_0$, $\tau_i = \tilde{\tau}_i, i = 1, \dots, m-1$ and τ_m (defined as in (26)) satisfy all the conditions of Theorem 6.1 and hence the corresponding closed-loop system is pendulum-like and Lagrange stable.

VII. ILLUSTRATIVE EXAMPLE

To illustrate the theory developed in this paper, we consider a system consisting of three connected pendulums, as shown in Figure 2, where the pendulums are connected using torsional springs and both pendulums and springs are supported by a rigid ring. The pendulums oscillate in planes perpendicular to the ring and the torsional torque of the springs obeys the angular form of the Hooke's law $F = -k\Delta\theta$, where $\Delta\theta$ is the angular displacement, F is the spring torque and k is the torque constant. This system can be considered as a prototype of many applications such as power systems, mechanical systems, network systems, etc. Therefore, the Lagrange stabilization of this system suggests many potential applications of the proposed method. Suppose that the measurements consist of the angular velocity of a pendulum and the angular difference between any two neighboring pendulums. As a result, all absolute positions of the pendulums are unobservable. Also, our A matrix has a single zero eigenvalue which is an unobservable mode of the system. Hence, Assumption 4.1 is satisfied. Let $x_1 = \theta_1, x_2 = \dot{\theta}_1, x_3 = \theta_2, x_4 = \dot{\theta}_2, x_5 = \theta_3$ and $x_6 = \dot{\theta}_3$. Then,

the system can be described by the state equations of the form (5) with the following matrices and nonlinearities

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ k_1 + k_3 & -\alpha_1 & -k_1 & 0 & -k_3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -k_1 & 0 & k_1 + k_2 & -\alpha_2 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k_3 & 0 & -k_2 & 0 & k_2 + k_3 & -\alpha_3 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_1 = \beta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad D_{12} = \varepsilon_1 I_3, \\ C_2 &= \gamma \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \\ D_{21} &= \varepsilon_2 I_3, \quad \text{and} \quad \phi(z) = [\sin z_1 \quad \sin z_2 \quad \sin z_3]^T. \quad (30) \end{aligned}$$

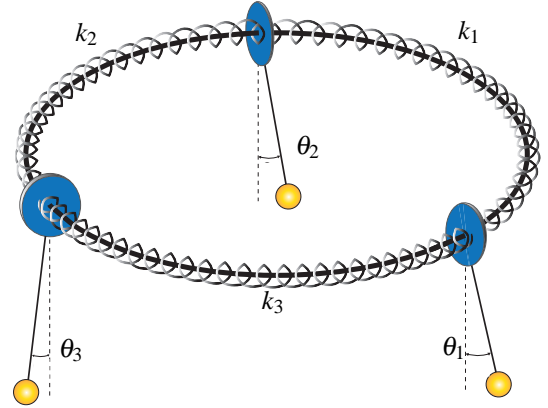


Fig. 2. A system of three pendulums connected on a ring by torsional springs.

Note that this system has multiple nonlinearities and thus the results of [3], [11]–[13] cannot be applied. Also, the nonlinearities do not have the special structure required in [6] to apply the result of that paper.

The damping coefficients are $\alpha_1 = 0.1$, $\alpha_2 = 0.05$, $\alpha_3 = 0.08$. The torque constants are $k_1 = 0.02$, $k_2 = 0.03$, $k_3 = 0.05$. Also, we specify the constants $\beta = 0.2$, $\gamma = 0.5$, $\varepsilon_1 = \varepsilon_2 = 0.1$. It is easy to verify that the system (5), (30) satisfies Assumption 4.1. Also, all of the coefficients of the system (5), (30) are rational. We choose $\tau_0 = 2\pi$ to ensure that Assumption 4.2 is satisfied (T will have rational elements in this case). Therefore, Theorem 4.1 is applicable to the system. Choosing $\tau_1 = 0.4$, $\tau_2 = 0.6$, $\tau_3 = 0.5$ and $\lambda = 0.5$ and solving the Riccati equations (18) and (19) gives solutions which satisfy all of the conditions of Theorem 4.1. Therefore, the solution to Problem 1 for the system (5), (30) can be constructed using this theorem. To illustrate the fact that the resulting controller is such that the closed-loop system is Lagrange stable, a series of simulations has been carried out with different initial values. These simulations have confirmed that the trajectories of the closed-loop system are bounded. This can be seen in Figure

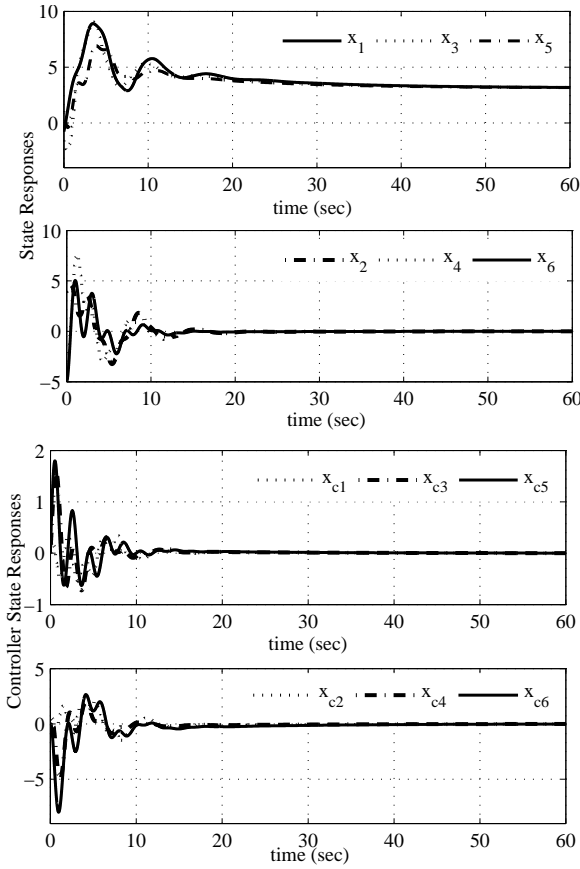


Fig. 3. System state responses and controller state responses of the closed-loop system for initial values $[x_1(0), \dots, x_6(0)]^T = [-\frac{\pi}{4}, 4, -\frac{\pi}{2}, -3, \frac{\pi}{3}, -5]$ and $x_{ci} = 0, i = 1, \dots, 6$.

3, which shows the state responses of the system and the controller state responses for one set of initial conditions, when the output feedback controller is applied. In addition, our simulations reveal that the trajectories of the closed-loop system converge. Using Theorem 1 in [17] and the results in [1], it can be verified that the closed-loop system has the property of dichotomy and the gradient-like property, which explains the observed convergence.

VIII. CONCLUSIONS AND FUTURE RESEARCH

This paper has studied the Lagrange stabilization problem for nonlinear systems with multiple nonlinearities. In order to facilitate the controller synthesis for these systems, a pseudo- H_∞ control theory is developed. Sufficient conditions for the solution to state feedback and output feedback pseudo- H_∞ control problems are given. However, corresponding necessary conditions are yet to be obtained. The pseudo- H_∞ control theory is applied to solve output feedback and state feedback Lagrange stabilization problems for nonlinear systems with multiple nonlinearities. The efficacy of the method is illustrated by an example involving coupled nonlinear pendulums on a ring.

This paper has considered the case where the nonlinear system contains decoupled nonlinearities. That is, as illustrated in Figure 1, we consider independent scalar nonlinearity blocks each subject to a sector bound constraint. One possible area

for future research is to extend the approach of this paper to enable the consideration of nonlinear systems with coupled nonlinearities. This would involve allowing the nonlinear blocks in Figure 1 to have vector inputs and outputs and to replace the sector bounds by more general local quadratic constraints.

APPENDIX

A. Proof of Lemma 2.1

First note that $p_i \neq 0$ since $\Delta_i \neq 0$. From the conditions of the lemma, we have $E_i \bar{d} = \Delta_i \frac{q_i}{p_i}$. From (3) and the fact that $\frac{q_i}{p_i} \bar{p}$ is an integer, it follows that $\phi_i(t, C_i \bar{d} + C_i x) = \phi_i(t, \Delta_i \frac{q_i}{p_i} \bar{p} + C_i x) = \phi_i(t, C_i x)$. As $A \bar{d} = 0$, it follows that,

$$A(x + \bar{d}) + \sum_{i=1}^m B_i \phi_i(t, C_i \bar{d} + C_i x) = Ax + \sum_{i=1}^m \phi_i(t, C_i x), \quad (31)$$

for all x and t .

Consider an arbitrary solution $x(t, t_0, x_0)$ of the system (1), (2). Let $\bar{x}(t) = x(t, t_0, x_0) + \bar{d}$ for $t \geq t_0$. Then, $\bar{x}(t_0) = x_0 + \bar{d}$. Also, it readily follows from (31) that $\bar{x}(t) = x(t, t_0, x_0 + \bar{d})$. Furthermore, the local Lipschitz condition implies the uniqueness of this solution. Then, we have $\bar{x}(t) = x(t, t_0, x_0 + \bar{d}) = x(t, t_0, x_0) + \bar{d}$. Hence, the lemma follows. \square

B. An Outline of the Proof of Lemma 2.2:

Define $\mathcal{G}(x, \xi) \triangleq \sum_{i=1}^m \tau_i (-\mu_i^{-1} \xi_i - C_i x)^* (\mu_i^{-1} \xi_i - C_i x)$ where $\xi \in \mathcal{C}^m$ and $x \in \mathcal{C}^n$ are arbitrary complex vectors. Clearly, there exist constants $\delta > 0$ and $0 < v < 1$ such that

$$\begin{aligned} & \begin{bmatrix} \xi \\ \zeta \end{bmatrix}^* \begin{bmatrix} B^T \\ \left(\frac{\delta}{2v}\right)^{\frac{1}{2}} I \end{bmatrix} \\ & \times ((-j\omega - \lambda)I - A^T)^{-1} C^T M_\tau C ((j\omega - \lambda)I - A)^{-1} \\ & \times \begin{bmatrix} B^T \\ \left(\frac{\delta}{2v}\right)^{\frac{1}{2}} I \end{bmatrix}^T \begin{bmatrix} \xi \\ \zeta \end{bmatrix} - \xi^* M_\mu^{-1} M_\tau M_\mu^{-1} \xi - \zeta^* \zeta \\ & \leq -\frac{\delta}{2} (\xi^* M_\mu^{-1} M_\tau M_\mu^{-1} \xi + \zeta^* \zeta), \\ & \forall \omega \in \mathcal{R}, \forall [\xi^T \ \zeta^T]^T \in \mathcal{C}^{m+n}. \end{aligned} \quad (32)$$

Given $\omega \in \mathcal{R}$, we define

$$\bar{\sigma} = ((j\omega - \lambda)I - A)^{-1} [B \left(\frac{\delta}{2v}\right)^{\frac{1}{2}} I] \bar{\xi}$$

and

$$\mathcal{G}_a(\bar{\sigma}, \bar{\zeta}) = \bar{\sigma}^* C^T M_\tau C \bar{\sigma} - \bar{\zeta}^* M_a \bar{\zeta},$$

where $\bar{\zeta} = \begin{bmatrix} \xi \\ \zeta \end{bmatrix}$ and $M_a = \begin{bmatrix} M_\mu^{-1} M_\tau M_\mu^{-1} & 0 \\ 0 & I \end{bmatrix}$. Therefore, it follows from (32) that

$$\mathcal{G}_a(\bar{\sigma}, \bar{\zeta}) \leq -\frac{\delta}{2} \bar{\zeta}^* M_a \bar{\zeta}, \quad \forall \omega \in \mathcal{R}, \bar{\zeta} \in \mathcal{C}^{m+n}. \quad (33)$$

Furthermore, since M_a is a positive definite matrix, the inequality (33) implies that $\mathcal{G}_a(\bar{\sigma}, \bar{\zeta}) < 0$, for all $\bar{\zeta} \in \mathcal{C}^{m+n}$ such that $\|\bar{\zeta}\| \neq 0$. Also, the pair $(A, [B \sqrt{\frac{\delta}{2v}} I_{n \times n}])$ is controllable.

Using Theorem 1.11.1 in [1], it follows that there exists a Hermitian matrix $P = P^*$ satisfying $2x^*P((A + \lambda I)x + B\xi + \sqrt{\frac{\delta}{2v}}\zeta) + \bar{\sigma}^*C^TM_\tau C\bar{\sigma} - \xi^*M_\mu^{-1}M_\tau M_\mu^{-1}\xi - \zeta^*\zeta < 0$, for all $x \in \mathcal{C}^n$, $\bar{\xi} = [\xi^T \ \zeta^T]^T \in \mathcal{C}^{m+n}$ such that $\|x\| + \|\bar{\xi}\| + \|\zeta\| \neq 0$. Letting $\zeta = 0$, this implies that there exists an $n \times n$ matrix $P = P^T$ such that

$$2x^*P[Ax + B\xi] < -2\lambda x^*Px - \mathcal{G}(x, \xi) \quad (34)$$

for all $x \in \mathcal{C}^n$, $\xi \in \mathcal{C}^m$ such that $\|x\| + \|\xi\| \neq 0$. Letting $\xi = 0$ in (34), we obtain that there exists a $r > 0$ such that $2x^*P[A + \lambda I]x < -rx^Tx < 0$.

Note that the pair $(A + \lambda I, rI)$ is observable. Since the matrix $A + \lambda I$ is pseudo-Hurwitz, then using Theorem 3 in [18] gives that P is pseudo-positive definite.

In a similar way to the proof of Theorem 2.6.1 in [1], we can prove that the set $\{x \in \mathcal{R}^n : x^TPx < 0\}$ is positively invariant for the nonlinear system (1), (2), (3) and further prove that the solution $x(t, t_0, x_0)$ of the system (1), (2), (3) is bounded. \square

C. Proof of Theorem 3.1:

In order to prove Theorem 3.1, some preliminary results are required.

Lemma A.1: Suppose the pair (C, A) has no unobservable modes on the $j\omega$ axis. If the Lyapunov equation $A^TP + PA + C^TC = 0$ has a pseudo-positive definite solution $P = P^T$, then the matrix A is pseudo-Hurwitz.

In order to prove Lemma A.1, we require the following results:

Lemma A.2 ([19]): Let \bar{P} be a symmetric matrix of the form $\bar{P} = \begin{bmatrix} 0 & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_{22} \end{bmatrix}$, where $\bar{P}_{22} = \bar{P}_{22}^T$ and \bar{P}_{12}^T are $n_2 \times n_2$ and $n_2 \times n_1$ matrices, respectively. Also, let $k \triangleq \text{rank}(\bar{P}_{12}^T)$. Then

$$\text{In}(\bar{P}) = \text{In}(\bar{P}_{22}/\ker(\bar{P}_{12})) + (k, k, n_1 - k), \quad (35)$$

where $\ker(\bar{P}_{12}) = \{\zeta \in \mathbb{R}^m : \bar{P}_{12}\zeta = 0\}$ and $\bar{P}_{22}/\ker(\bar{P}_{12})$ represents the restriction of \bar{P}_{22} to $\ker(\bar{P}_{12})$.

Lemma A.3 ([20]): If $A \in \mathcal{R}^{n \times n}$ and if $\lambda, \mu \in \sigma(A)$ are eigenvalues of A where $\lambda \neq \mu$, then any left eigenvector of A corresponding to μ is orthogonal to any right eigenvector of A corresponding to λ .

Proof of Lemma A.1: The Kalman decomposition [16] establishes the existence of a matrix T which transforms the matrix pair (A, C) into the form $\bar{A} = TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}$, $\bar{C} = CT^{-1} = [0 \ \bar{C}_2]$ where the pair $(\bar{A}_{22}, \bar{C}_2)$ is observable. The dimensions of the blocks in the above decomposition are as follows: $\bar{A}_{11} \in \mathcal{R}^{n_1 \times n_1}$, $\bar{A}_{12} \in \mathcal{R}^{n_1 \times n_2}$, $\bar{A}_{22} \in \mathcal{R}^{n_2 \times n_2}$, and the column dimension of \bar{C}_2 is n_2 . Correspondingly, let $\bar{P} = T^{-T}PT^{-1} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_{22} \end{bmatrix}$. It follows from the observability of $(\bar{A}_{22}, \bar{C}_2)$ that there exists a matrix \bar{K} such that $\delta(\bar{A} + \bar{K}\bar{C}) = 0$.

Using the equation $A^TP + PA + C^TC = 0$, it follows that

$$\begin{bmatrix} \bar{A}_{11}^T \bar{P}_{11} + \bar{P}_{11} \bar{A}_{11} \\ \bar{A}_{12}^T \bar{P}_{11} + \bar{A}_{22}^T \bar{P}_{12} + \bar{P}_{12}^T \bar{A}_{11} \\ \bar{P}_{11} \bar{A}_{12} + \bar{P}_{12} \bar{A}_{22} + \bar{A}_{11} \bar{P}_{12} \\ \bar{P}_{12}^T \bar{A}_{12} + \bar{A}_{12}^T \bar{P}_{12} + \bar{P}_{22} \bar{A}_{22} + \bar{A}_{22}^T \bar{P}_{22} + \bar{C}_2^T \bar{C}_2 \end{bmatrix} = 0. \quad (36)$$

Hence,

$$\bar{A}_{11}^T \bar{P}_{11} + \bar{P}_{11} \bar{A}_{11} = 0. \quad (37)$$

Claim 1: If the pair (\bar{C}, \bar{A}) is such that there exists a matrix \bar{K} satisfying $\delta(\bar{A} + \bar{K}\bar{C}) = 0$, then $\text{Re}\lambda(\bar{A}_{11}) \neq 0$ for $\forall \lambda \in \sigma(\bar{A}_{11})$.

To establish Claim 1, we rewrite \bar{K} as $\bar{K} = \begin{bmatrix} \bar{K}_1 \\ \bar{K}_2 \end{bmatrix}$. Then $\bar{A} + \bar{K}\bar{C} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} + \bar{K}_1 \bar{C}_2 \\ 0 & \bar{A}_{22} + \bar{K}_2 \bar{C}_2 \end{bmatrix}$. If there exists an eigenvalue of \bar{A}_{11} such that $\text{Re}\lambda(\bar{A}_{11}) = 0$, then $\bar{A} + \bar{K}\bar{C}$ obviously has purely imaginary eigenvalues. This contradicts the fact that \bar{K} is chosen so that $\delta(\bar{A} + \bar{K}\bar{C}) = 0$. Therefore, $\text{Re}\lambda(\bar{A}_{11}) \neq 0$. This completes the proof of the claim.

Combining Claim 1 and (37) gives that $\bar{P}_{11} = 0$. Also, the (1,2) block of (36) implies that

$$\bar{P}_{12} \bar{A}_{22} + \bar{A}_{11} \bar{P}_{12} = 0. \quad (38)$$

As \bar{P} is nonsingular, this implies that $\bar{P}_{12} \bar{P}_{12}^T > 0$. Applying Lemma A.2 to \bar{P} gives that $\text{In} \bar{P} = \text{In}(\bar{P}_{22}/\ker \bar{P}_{12}) + (\text{rank} \bar{P}_{12}^T, \text{rank} \bar{P}_{12}^T, n_1 - \text{rank} \bar{P}_{12}^T)$. It is known that $\text{In} \bar{P} = (n-1, 1, 0)$. This implies that

- 1) $0 = \delta(\bar{P}) = \delta(\bar{P}_{22}/\ker \bar{P}_{12}) + (n_1 - \text{rank} \bar{P}_{12}^T)$. For $\bar{P}_{12}^T \in \mathcal{R}^{n_2 \times n_1}$, it always holds that $\text{rank} \bar{P}_{12}^T \leq n_1$. Then, $\delta(\bar{P}_{22}/\ker \bar{P}_{12}) \leq 0$. So, $\delta(\bar{P}_{22}/\ker \bar{P}_{12}) = 0$ holds. This further implies that $n_1 = \text{rank} \bar{P}_{12}^T$. Also, the condition $\delta(\bar{P}_{22}/\ker \bar{P}_{12}) = 0$ implies that the matrix $\bar{P}_{22}/\ker \bar{P}_{12}$ is nonsingular and has no purely imaginary eigenvalues.
- 2) $1 = v(\bar{P}) = v(\bar{P}_{22}/\ker \bar{P}_{12}) + \text{rank} \bar{P}_{12}^T$. Hence, $\text{rank} \bar{P}_{12}^T \leq 1$ and $\bar{P}_{12} \bar{P}_{12}^T > 0$ imply $\text{rank} \bar{P}_{12}^T = 1$ and $v(\bar{P}_{22}/\ker \bar{P}_{12}) = 0$. Hence, $\bar{P}_{22}/\ker \bar{P}_{12} \triangleq S^T \bar{P}_{22} S$ is symmetric and positive definite, where the columns of S form a basis for $\ker \bar{P}_{12}$.
- 3) Finally, the identity $\pi(\bar{P}) = n-1 = \pi(\bar{P}_{22}/\ker \bar{P}_{12}) + \text{rank} \bar{P}_{12}^T$ implies $\pi(\bar{P}_{22}/\ker \bar{P}_{12}) = n-2$.

Therefore, it follows that $n_1 = 1$. Hence \bar{A}_{11} is a scalar, \bar{P}_{12} is a row vector of dimension $n-1$, and \bar{P}_{22} is a $(n-1) \times (n-1)$ matrix. The dimension of $S^T \bar{P}_{22} S$ is equal to $n-2$.

As $\bar{P}_{12} \in \mathcal{R}^{1 \times (n-1)}$, $\bar{P}_{12} \neq 0$, (38) implies that $-\bar{A}_{11}$ is an eigenvalue of \bar{A}_{22} and \bar{P}_{12} is the corresponding left eigenvector. Now, let λ be any eigenvalue of \bar{A}_{22} such that $\lambda \neq -\bar{A}_{11}$ and let q be a corresponding right eigenvector; that is, $\bar{A}_{22}q = \lambda q$. Then Lemma A.3 implies that $\bar{P}_{12}q = 0$. Hence, $q \in \text{Ker} \bar{P}_{12}$.

Pre- and post-multiplying the (2,2) block of (36) by q^T and q respectively implies that $q^T \bar{P}_{22} \bar{A}_{22} q + q^T \bar{A}_{22}^T \bar{P}_{22} q + q^T \bar{C}_2^T \bar{C}_2 q = 0$. Therefore, $\lambda q^T \bar{P}_{22} q + \bar{\lambda} q^T \bar{P}_{22} q + \|\bar{C}_2 q\|^2 = 0$.

Using the fact that $\alpha = q^T \bar{P}_{22} q$ is positive on $\text{Ker} \bar{P}_{12}$, we have $2\alpha \text{Re}(\lambda) + \|\bar{C}_2 q\|^2 = 0$. Since $q \neq 0$ is an eigenvector of \bar{A}_{22} , $\bar{C}_2 q \neq 0$. Therefore, $\text{Re}(\lambda) < 0$.

The above derivation shows that all eigenvalues of \bar{A}_{22} , possibly with the exception of $-\bar{A}_{11}$, have negative real part.

Therefore, if $-\bar{A}_{11}$ is negative, then \bar{A}_{22} is Hurwitz; if $-\bar{A}_{11}$ is positive, then \bar{A}_{22} has all the eigenvalues $\lambda \neq -\bar{A}_{11}$ negative except $-\bar{A}_{11}$.

Now, we can conclude that the spectrum of \bar{A} is $\sigma(\bar{A}) = \{-\bar{A}_{11}, \bar{A}_{11} \text{ and } \lambda : \lambda \neq -\bar{A}_{11}, \operatorname{Re} \lambda < 0\}$. Also, since the pair (C, A) has no unobservable modes on the imaginary axis, it follows that $\bar{A}_{11} \neq 0$. Hence, \bar{A} is pseudo-Hurwitz. This completes the proof of Lemma A.1. \square

Proof of Theorem 3.1: By assumption, P is such that $\delta(A + BB^T P) = 0$. Letting $\bar{C} = \begin{bmatrix} B^T P \\ C \end{bmatrix}$, $\bar{K} = \begin{bmatrix} B & 0 \end{bmatrix}$, it follows that $A + \bar{K}\bar{C}$ is such that $\delta(A + \bar{K}\bar{C}) = 0$. Therefore, $(A + \bar{K}\bar{C}, \bar{C})$ has no unobservable mode on the imaginary axis and hence (A, \bar{C}) has no unobservable mode on the imaginary axis, either. Applying Lemma A.1 to the Lyapunov equation $A^T P + PA + \bar{C}^T \bar{C} = 0$, it follows that A is pseudo-Hurwitz. Hence, $\det(j\omega I - A) \neq 0$ for all $\omega \in \mathcal{R}$.

Now, we show that (7) holds. Since A is pseudo-Hurwitz, then $\det(j\omega - A) \neq 0$, $\forall \omega \in \mathcal{R}$. Hence, (8) implies that

$$\begin{aligned} G(-j\omega)^T G(j\omega) &= \\ I - [I - B^T P(-j\omega I - A)^{-1} B]^T [I - B^T P(j\omega I - A)^{-1} B] \\ &\leq I \end{aligned} \quad (39)$$

for all $\omega \geq 0$. It follows that $\max_{\omega \in \mathcal{R}} \{\sigma_{\max}[G(j\omega)G(-j\omega)^T]\} \leq 1$. Furthermore, note that $G(j\omega) \rightarrow 0$ as $\omega \rightarrow \infty$. Now suppose that there exists an $\bar{\omega} \geq 0$ such that $\max_{\omega} \{\sigma_{\max}[G(-j\omega)^T G(j\omega)]\} = 1$. It follows from (39) that there exists a vector z such that $[I - B^T P(j\bar{\omega} - A)^{-1} B]z = 0$. Hence, $\det[I - B^T P(j\bar{\omega} - A)^{-1} B] = 0$. However, using a standard result on determinants, it follows that $\det[j\bar{\omega}I - A - BB^T P] = \det[j\bar{\omega}I - A] \det[I - B^T P(j\bar{\omega}I - A)^{-1} B]$. Thus $\det[j\bar{\omega}I - A - BB^T P] = 0$. This conclusion contradicts the assumption that $\delta(A + BB^T P) = 0$. Hence, (7) holds. \square

D. Proof of Theorem 3.2:

$$\text{Let } \mu \triangleq \left(\max_{\omega \in \mathcal{R}} \{\sigma_{\max}[(-j\omega I - A^T)^{-1} C^T C(j\omega I - A)^{-1}]\} \right)^{\frac{1}{2}}.$$

It follows from (7) that there exist an $\varepsilon \geq 0$ such that $G(j\omega)G(j\omega)^T \leq (1 - \varepsilon)I$. Hence, $\frac{\varepsilon}{2\mu^2} C(j\omega I - A)^{-1} (-j\omega I - A^T)^{-1} C^T \leq \frac{\varepsilon}{2} I$ for all $\omega \geq 0$. Then, given any $\omega \geq 0$, $C(j\omega I - A)^{-1} \tilde{B} \tilde{B}^T (-j\omega I - A^T)^{-1} C^T \leq (1 - \frac{\varepsilon}{2}) I$, where \tilde{B} is a non-singular matrix defined by $\tilde{B} \tilde{B}^T = BB^T + \varepsilon/2\mu^2 I$. This further implies that

$$\tilde{B}^T (-j\omega I - A^T)^{-1} C^T C(j\omega I - A)^{-1} \tilde{B} \leq \left(1 - \frac{\varepsilon}{2}\right) I \quad (40)$$

for all $\omega \geq 0$. Let $\eta^2 \triangleq \max_{\omega \in \mathcal{R}} \sigma_{\max}[\tilde{B}^T (-j\omega I - A^T)^{-1} C^T C(j\omega I - A)^{-1} \tilde{B}]$. Hence, $\frac{\varepsilon}{2\eta^2} \tilde{B}^T (-j\omega I - A^T)^{-1} (j\omega I - A)^{-1} \tilde{B} \leq \frac{\varepsilon}{2} I$, holds for all $\omega \geq 0$. From (40), it follows that given any $\omega \geq 0$,

$$\tilde{G}(-j\omega)^T \tilde{G}(j\omega) \leq I \quad (41)$$

where $\tilde{G}(s) = \tilde{C}(sI - A)^{-1} \tilde{B}$ with \tilde{C} being a non-singular matrix defined so that $\tilde{C}^T \tilde{C} = C^T C + (\varepsilon/2\eta^2)I$. Furthermore, (41) implies $\tilde{G}(j\omega)\tilde{G}(-j\omega)^T \leq I$.

Since A has no eigenvalue on the $j\omega$ -axis and the pair (A, \tilde{B}) is stabilizable (since it is controllable), it follows from Theorem 13.34 in [10] and (41) that there exists a right coprime factorization $\tilde{G}(s) = \tilde{N}(s)\tilde{M}^{-1}(s)$ such that $\tilde{M}(s) \in \mathcal{RH}_{\infty}$ is an inner transfer function matrix where $\tilde{M}(s) = \tilde{F}(sI - A - \tilde{B}\tilde{F})^{-1} \tilde{B} + I \in \mathcal{RH}_{\infty}$, $\tilde{N}(s) = \tilde{C}(sI - A - \tilde{B}\tilde{F})^{-1} \tilde{B} \in \mathcal{RH}_{\infty}$ with $\tilde{F} = -\tilde{B}^T \tilde{X}$, and the Riccati equation $A^T \tilde{X} + \tilde{X}A - \tilde{X}\tilde{B}\tilde{B}^T \tilde{X} = 0$ has a solution $\tilde{X} \geq 0$ such that $A - \tilde{B}\tilde{B}^T \tilde{X}$ is stable. Since $\tilde{M}(s)$ is an inner transfer function, it follows that $\tilde{N}(j\omega)\tilde{N}^T(-j\omega) = \tilde{G}(j\omega)\tilde{G}^T(-j\omega) \leq I$. Applying the bounded real lemma (e.g., see [7]), the above condition is equivalent to the existence of a stabilizing solution to the Riccati equation

$$(A - \tilde{B}\tilde{B}^T \tilde{X})^T \tilde{P} + \tilde{P}(A - \tilde{B}\tilde{B}^T \tilde{X}) + \tilde{P}\tilde{B}\tilde{B}^T \tilde{P} + \tilde{C}^T \tilde{C} = 0. \quad (42)$$

Let $\tilde{P} = \tilde{P} - \tilde{X}$. Then substituting this into (42) gives that

$$\begin{aligned} (A^T \tilde{X} + \tilde{X}A - \tilde{X}^T \tilde{B}\tilde{B}^T \tilde{X}) \\ + (A^T \tilde{P} + \tilde{P}A + \tilde{P}\tilde{B}\tilde{B}^T \tilde{P} + \tilde{C}^T \tilde{C}) = 0. \end{aligned} \quad (43)$$

Therefore, (43) implies that $A^T \tilde{P} + \tilde{P}A + \tilde{P}\tilde{B}\tilde{B}^T \tilde{P} + \tilde{C}^T \tilde{C} + \frac{\varepsilon}{2\mu^2} \tilde{P}^2 + \frac{\varepsilon}{2\eta^2} I = 0$. This implies that $P = \tilde{P}$ satisfies (9). This proves the first claim of the theorem. Now we prove the second claim.

From (7), it follows that $G(j\omega)G^T(-j\omega) \leq I$. As the pair (A, B) is stabilizable, Theorem 13.34 in [10] implies that there exists a right coprime factorization $G(s) = N(s)M^{-1}(s)$ such that $M(s) \in \mathcal{RH}_{\infty}$ is an inner transfer function matrix where $M(s) = F(sI - A - BF)^{-1} B + I \in \mathcal{RH}_{\infty}$, $N(s) = C(sI - A - BF)^{-1} B \in \mathcal{RH}_{\infty}$ with $F = -B^T X$, and the Riccati equation $A^T X + XA - XBB^T X = 0$ has a solution $X \geq 0$ such that $A - BB^T X$ is stable. Since $M(s)$ is an inner transfer function, it follows that $N(j\omega)N^T(-j\omega) = G(j\omega)G^T(-j\omega) \leq I$. Applying the bounded real lemma [7], the above condition is equivalent to the condition that the following Riccati equation has a stabilizing solution

$$(A - BB^T X)^T \bar{P} + \bar{P}(A - BB^T X) + \bar{P}BB^T \bar{P} + C^T C = 0. \quad (44)$$

Let $P = \bar{P} - X$. Then substituting this into (45) gives that

$$(A^T X + XA - X^T BB^T X) + (A^T P + PA + PBB^T P + C^T C) = 0. \quad (45)$$

Therefore, the Riccati equation (8) has a stabilizing solution. Furthermore, as the pair (A, C) is observable, it follows from the Inertia theorem in [21] that the solution $P = P^T$ of the Riccati equation (8) is a pseudo-positive definite matrix. This completes the proof. \square

Proof of Theorem 3.3: The Riccati equation (10) can be written as

$$\begin{aligned} (A - B_2 E_1^{-1} D_{12}^T C_1)^T P + P(A - B_2 E_1^{-1} D_{12}^T C_1) \\ + P B_1 B_1^T P + C_1^T (I - D_{12} E_1^{-1} D_{12}^T) C_1 \\ - P B_2 E_1^{-1} D_{12}^T (I - D_{12} E_1^{-1} D_{12}^T) C_1 - P B_2 E_1^{-1} B_2^T P \\ - C_1^T (I - D_{12} E_1^{-1} D_{12}^T) D_{12} E_1^{-1} B_2^T P = 0. \end{aligned} \quad (46)$$

As the Riccati equation (10) has a solution $P = P^T$ which is pseudo-positive definite, the equation (46) also has this

property. Substituting $K = -E_1^{-1}(D_{12}^T C_1 + B_2^T P)$ into (46) implies that

$$(A + B_2 K)^T P + P(A + B_2 K) + P B_1 B_1^T P + (C_1 + D_{12} K)^T (C_1 + D_{12} K) = 0 \quad (47)$$

has a solution $P = P^T$ which is pseudo-positive definite. Also, the fact that the matrix (11) has no purely imaginary eigenvalues implies that $A + B_2 K + B_1 B_1^T P$ has no purely imaginary eigenvalues. Therefore, it follows from Theorem 3.1 that the resulting closed-loop system

$$\begin{aligned} \dot{x} &= (A - B_2 E_1^{-1} D_{12}^T C_1 - B_2 E_1^{-1} B_2^T P)x + B_1 w, \\ z &= C_1 - D_{12} E_1^{-1} (D_{12}^T C_1 + B_2^T P)x \end{aligned}$$

is pseudo strict bounded real. This completes the proof of Theorem 3.3. \square

E. Proof of Theorem 3.4

In order to prove Theorem 3.4, the following lemma is introduced.

Lemma A.4: Suppose the conditions of Theorem 3.4 hold. Then, the matrix $Z \triangleq (I - YX)^{-1}Y = Y(I - XY)^{-1} > 0$ is a stabilizing solution to the Riccati equation

$$A_* Z + Z A_*^T - Z M_* Z + N_* = 0 \quad (48)$$

where $A_* = A - B_1 D_{21}^T E_2^{-1} C_2 + B_1 (I - D_{21}^T E_2^{-1} D_{21}) B_1^T X$, $N_* = B_1 (I - D_{21}^T E_2^{-1} D_{21} B_1^T)$, $M_* = (C_2 + D_{21} B_1^T X)^T E_2^{-1} (C_2 + D_{21} B_1^T X) - (B_2^T X + D_{12}^T C_1)^T E_1^{-1} (B_2^T X + D_{12}^T C_1)$.

The proof of this lemma is similar to that of Lemma 3.2 in [8] and is omitted.

Proof of Theorem 3.4: We will prove that the compensator of the form (6), (15) makes the closed-loop system pseudo strict bounded real. In order to establish this fact, note that Lemma A.4 implies that matrix $Z = (I - YX)^{-1}Y > 0$ is a stabilizing solution to the Riccati equation (48). Substituting $(I - YX)^{-1}Y = Z$ and $(I - YX)^{-1} = (I + ZX)$ into (15), it follows that the compensator input matrix B_c can be written as

$$B_c = B_1 D_{21}^T E_2^{-1} + Z(C_2^T + X B_1 D_{21}^T) E_2^{-1}. \quad (49)$$

We now form the closed-loop system associated with system (5) and compensator (6). This system is described by the state equation

$$\begin{aligned} \dot{\eta} &= \bar{A} \eta + \bar{B} w, \\ z &= \bar{C} \eta, \end{aligned} \quad (50)$$

where $\eta \triangleq \begin{bmatrix} x \\ x - x_c \end{bmatrix}$, $\bar{A} \triangleq \begin{bmatrix} A + B_2 C_c & -B_2 C_c \\ A - A_c + B_2 C_c - B_c C_2 & A_c - B_2 C_c \end{bmatrix}$, $\bar{B} \triangleq \begin{bmatrix} B_1 \\ B_1 - B_c D_{21} \end{bmatrix}$ and $\bar{C} \triangleq \begin{bmatrix} C_1 + D_{12} C_c & -D_{12} C_c \end{bmatrix}$.

In order to verify that this system is pseudo strict bounded real, we first recall that $Z > 0$ is a stabilizing solution to the Riccati equation (48). This implies that $Z > 0$ will also be a stabilizing solution to the Riccati equation

$$A_0 Z + Z A_0^T + Z C_0^T C_0 Z + B_0 B_0^T = 0 \quad (51)$$

where $A_0 \triangleq A - B_1 D_{21}^T E_2^{-1} C_2 + B_1 (I - D_{21}^T E_2^{-1} D_{21}) B_1^T X - Z(C_2 + D_{21} B_1^T X)^T E_2^{-1} (C_2 + D_{21} B_1^T X)$, $B_0 \triangleq B_1 (I - D_{21}^T E_2^{-1} D_{21}) - Z(C_2 + D_{21} B_1^T X)^T E_2^{-1} D_{21}$, $C_0 \triangleq E_1^{-1} (B_2^T X + D_{12}^T C_1)$.

Let $W = Z^{-1} > 0$, then the Riccati equation (51) leads to

$$A_0^T W + W A_0 + W B_0 B_0^T W + C_0^T C_0 = 0. \quad (52)$$

Now, we prove that $W = W^T$ is an anti-stabilizing solution of (52). Using the Riccati equation (52), it follows that $-(A_0 + Z C_0^T C_0) = Z(A_0^T + Z^{-1} B_0 B_0^T) Z^{-1}$. Hence the matrix $-(A_0^T + Z C_0^T C_0)$ is similar to the matrix $(A_0 + B_0 B_0^T W)^T$. Since Z is a stabilizing solution to (51), the matrix $A_0^T + Z C_0^T C_0$ must be Hurwitz and hence the matrix $A_0 + B_0 B_0^T W$ must be anti-Hurwitz; i.e., W is an anti-stabilizing solution to (52).

Now, we define $\Sigma \triangleq \begin{bmatrix} X & 0 \\ 0 & W \end{bmatrix}$. As X is pseudo-positive definite and $W > 0$, it follows that Σ is also pseudo-positive definite. Using equations (13), (15), (49), (52), it is straightforward to verify that Σ satisfies the Riccati equation $\bar{A}^T \Sigma + \Sigma \bar{A} + \Sigma \bar{B} \bar{B}^T \Sigma + \bar{C}^T \bar{C} = 0$. Furthermore, it is straightforward to verify that $\bar{A} + \bar{B} \bar{B}^T \Sigma = \begin{bmatrix} \check{A}_{11} & \check{A}_{12} \\ 0 & A_0 + B_0 B_0^T W \end{bmatrix}$ where $\check{A}_{11} = A - B_2 E_1^{-1} D_{12}^T C_1 - (B_2 E_1^{-1} B_2^T - B_1 B_1^T) X$, $\check{A}_{12} = B_2 E_1^{-1} B_2^T + B_2 E_1^{-1} D_{12}^T C_1 + B_1 (I - D_{21}^T E_2^{-1} D_{21}) B_1^T W - B_1 D_{21}^T E_2^{-1} (C_2 + D_{21} B_1^T X) Z W$. Using the fact that X is a stabilizing solution to (13) and W is an anti-stabilizing solution to (52), it follows that $\bar{A} + \bar{B} \bar{B}^T \Sigma$ has no purely imaginary eigenvalues. We have noted previously that the matrix Σ is pseudo-positive definite. Therefore, using Theorem 3.1, we conclude that the system (50) is pseudo strict bounded real. Using the fact that $\eta = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix}$, it follows that the closed-loop system

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} &= \begin{bmatrix} A & B_c C_c \\ B_c C_2 & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} B_1 \\ B_c D_{21} \end{bmatrix} w, \\ z &= \begin{bmatrix} C_1 & D_{12} C_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} \end{aligned}$$

is also pseudo strict bounded real. This completes the proof of Theorem 3.4. \square

F. Proof of Theorem 3.5:

Consider the system described by the state equations

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A} \tilde{x} + \tilde{B}_2 \tilde{u} + \tilde{B}_1 \tilde{w}, \\ \tilde{z} &= \tilde{C}_1 \tilde{x} + \tilde{D}_{12} \tilde{u}, \\ \tilde{y} &= \tilde{C}_2 \tilde{x} + \tilde{D}_{21} \tilde{w}, \end{aligned} \quad (53)$$

where

$$\begin{aligned} \tilde{A} &= A^T, \quad \tilde{B}_1 = C_1^T, \quad \tilde{B}_2 = C_2^T, \quad \tilde{C}_1 = B_1^T, \quad \tilde{D}_{12} = D_{21}^T, \\ \tilde{C}_2 &= B_2^T, \quad \tilde{D}_{21} = D_{12}^T, \end{aligned} \quad (54)$$

Let

$$\tilde{E}_1 = \tilde{D}_{12}^T \tilde{D}_{12} = E_2, \quad \tilde{E}_2 = \tilde{D}_{21}^T \tilde{D}_{21} = E_1, \quad \tilde{X} = Y, \quad \tilde{Y} = X. \quad (55)$$

Substituting the matrices in (54) and (55) into Conditions (i), (ii), (iii) of the theorem gives that the system (53) satisfies the following conditions of Theorem 3.4:

- (i') The Riccati, as shown below, has a pseudo-positive definite stabilizing solution

$$\begin{aligned} & (\tilde{A} - \tilde{B}_2 \tilde{E}_1^{-1} \tilde{D}_{12}^T \tilde{C}_1)^T \tilde{X} + \tilde{X} (\tilde{A} - \tilde{B}_2 \tilde{E}_1^{-1} \tilde{D}_{12}^T \tilde{C}_1) \\ & + \tilde{X} (\tilde{B}_1 \tilde{B}_1^T - \tilde{B}_2 \tilde{E}_1^{-1} \tilde{B}_2^T) \tilde{X} \\ & + \tilde{C}_1^T (I - \tilde{D}_{12} \tilde{E}_1^{-1} \tilde{D}_{12}^T) \tilde{C}_1 = 0. \end{aligned} \quad (56)$$

- (ii') The following Riccati equation has a positive definite stabilizing solution

$$\begin{aligned} & (\tilde{A} - \tilde{B}_1 \tilde{D}_{21}^T \tilde{E}_2^{-1} \tilde{C}_2) \tilde{Y} + \tilde{Y} (\tilde{A} - \tilde{B}_1 \tilde{D}_{21}^T \tilde{E}_2^{-1} \tilde{C}_2)^T \\ & + \tilde{Y} (\tilde{C}_1^T \tilde{C}_1 - \tilde{C}_2^T \tilde{E}_2^{-1} \tilde{C}_2) \tilde{Y} \\ & + \tilde{B}_1 (I - \tilde{D}_{21}^T \tilde{E}_2^{-1} \tilde{D}_{21}) \tilde{B}_1^T = 0. \end{aligned} \quad (57)$$

- (iii') The matrix $\tilde{X}\tilde{Y}$ has a spectral radius strictly less than one, $\rho(\tilde{X}\tilde{Y}) < 1$.

Using Theorem 3.4, it follows that there exists a dynamic output feedback compensator of the form (6) such that the closed-loop system consisting of the system (53) and this compensator is pseudo strict bounded real. The parameters of this compensator are as follows:

$$\begin{aligned} \tilde{A}_c &= \tilde{A} + \tilde{B}_2 \tilde{C}_c - \tilde{B}_c \tilde{C}_2 + (\tilde{B}_1 - \tilde{B}_c \tilde{D}_{21}) \tilde{B}_1^T \tilde{X}, \\ \tilde{B}_c &= (I - \tilde{Y} \tilde{X})^{-1} (\tilde{Y} \tilde{C}_2^T + \tilde{B}_1 \tilde{D}_{21}^T) \tilde{E}_2^{-1}, \\ \tilde{C}_c &= -\tilde{E}_1^{-1} (\tilde{B}_2^T \tilde{X} + \tilde{D}_{12}^T \tilde{C}_1). \end{aligned} \quad (58)$$

Substituting the matrix in (54) and (55) into (58), the transfer function of this closed-loop system becomes $\tilde{G}(s) = [B_1^T \ D_{21}^T B_c^T] \left(sI - \begin{bmatrix} A^T & C_2^T B_c^T \\ C_c^T B_2^T & A_c^T \end{bmatrix} \right)^{-1} \begin{bmatrix} C_1^T \\ C_c^T D_{12}^T \end{bmatrix}$.

Consider the system (5) with compensator (6) whose parameters are determined by (16). It is readily seen that the transfer function of this closed-loop system $\check{G}(s)$ satisfies $\check{G}(s) = \tilde{G}^T(s)$. Therefore, from the fact that the system (53), (54), (55), (58) is pseudo strict bounded real, it follows that $\max_{\omega} \sigma_{\max}[\check{G}^T(-j\omega)\check{G}(j\omega)] < 1$. Also, $\begin{bmatrix} \tilde{A} & \tilde{B}_2 \tilde{C}_c \\ \tilde{B}_c \tilde{C}_2 & \tilde{A}_c \end{bmatrix}^T = \begin{bmatrix} A & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix}$ and is pseudo-Hurwitz. Hence, the closed-loop system (5), (6), (16) is pseudo strict bounded real. \square

G. Proof of Theorem 4.1:

We first prove that the closed-loop system

$$\begin{aligned} \begin{bmatrix} \dot{x}_c \\ \dot{x} \end{bmatrix} &= \begin{bmatrix} A_c & B_c C_2 \\ B_2 C_c & A \end{bmatrix} \begin{bmatrix} x_c \\ x \end{bmatrix} + \begin{bmatrix} B_c D_{21} \\ B_1 \end{bmatrix} w, \\ z &= [D_{12} C_c \ C_1] \begin{bmatrix} x_c \\ x \end{bmatrix}, \end{aligned} \quad (59)$$

obtained by substituting the controller (6), (20) into the system (5), is pendulum-like. Let $\tilde{d} = [0_{1 \times n} \ e_n^T T^T]^T$. Note the identity

$$\begin{aligned} & \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}^{-1} \begin{bmatrix} A_c & B_c C_2 \\ B_2 C_c & A \end{bmatrix} \tilde{d} \\ &= \begin{bmatrix} A_c & B_c C_{2a} & 0 \\ B_{2a} C_c & \tilde{A}_1 & 0 \\ B_{2b} C_c & \tilde{A}_2 & 0 \end{bmatrix} \begin{bmatrix} 0_{(2n-1) \times 1} \\ 1 \end{bmatrix} = 0. \end{aligned}$$

Since $\begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}$ is non-singular, it follows that $\begin{bmatrix} A_c & B_c C_2 \\ B_2 C_c & A \end{bmatrix} \tilde{d} = 0$. Using this fact and Assumption 4.1, it follows from Lemma 2.1 that the resulting closed-loop system (59) is pendulum-like system with respect to the set $\Pi(\tau_0 \bar{p} \tilde{d})$.

From the output feedback pseudo H_∞ control theory in Section III, Conditions I, II, III of the theorem imply that the matrix $\begin{bmatrix} \lambda I + A & B_c C_c \\ B_c C_2 & \lambda I + A_c \end{bmatrix}$ is pseudo-Hurwitz and the frequency-domain condition $\max_{\omega} \sigma_{\max}[\tilde{G}^T(-j\omega)\tilde{G}(j\omega)] < 1$ holds, where $\tilde{G}(\cdot)$ is defined as $\tilde{G}(s) \triangleq M_\tau^{-1} G_c(s) M_\tau^{-1}$ and here $G_c(s) \triangleq [C_1 \ D_{12} C_c] \left(s - \begin{bmatrix} \lambda I + A & B_c C_c \\ B_c C_2 & \lambda I + A_c \end{bmatrix} \right)^{-1} \begin{bmatrix} B_1 \\ B_c D_{21} \end{bmatrix}$. Then, it follows that $G_c^T(-j\omega) M_\tau G_c(j\omega) < M_\mu^{-1} M_\tau M_\mu^{-1}$ for all $\omega \in \mathcal{R}$. Now, all the conditions of Lemma 2.2 are satisfied and hence the closed-loop nonlinear system (59), (2), (3), (4) is Lagrange stable. \square

H. Proof of Theorem 5.1:

We first prove that the closed-loop system (59), obtained by applying the compensator (6), (22) to the system (5), is a pendulum-like system.

Since $\begin{bmatrix} I & 0 \\ 0 & \bar{T} \end{bmatrix}^{-1} \begin{bmatrix} A_c & B_c C_2 \\ B_2 C_c & A \end{bmatrix} \tilde{d} = \begin{bmatrix} A_c & B_c \tilde{C}_{2a} \\ \tilde{B}_{2a} & \tilde{A}_1 \end{bmatrix} \begin{bmatrix} B_c \tilde{C}_{2b} \\ \tilde{A}_2 \end{bmatrix} \begin{bmatrix} \tilde{d}_0 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} I & 0 \\ 0 & \bar{T} \end{bmatrix}$ is a non-singular matrix, it follows that $\begin{bmatrix} A_c & B_c C_2 \\ B_2 C_c & A \end{bmatrix} \tilde{d} = 0$. Using this fact and Condition IV of the theorem, it follows from Lemma 2.1 that the augmented closed-loop system (59), (2), (3), (4) is a pendulum-like system with respect to $\Pi(\bar{p} \tau_0 \tilde{d})$.

Using the output feedback pseudo H_∞ control theory given in Section III, it follows from Conditions I, II and III of the theorem that the closed-loop system (59) is pseudo strict bounded real. In a similar way to the proof of Theorem 4.1, we have $G_c^T(-j\omega - \lambda) M_\tau G_c(j\omega - \lambda) < M_\mu^{-1} M_\tau M_\mu^{-1}$. Now, using Lemma 2.2, it follows that the closed-loop system (59), (2), (3), (4) is Lagrange stable. \square

I. Proof of Theorem 5.2

The stabilizing solutions to the Riccati equations (18) and (19) are functions of the vector of constants $\bar{\tau}$. To highlight this, we use the notation $X(\bar{\tau})$ and $Y(\bar{\tau})$. In the proof of Theorem 5.2, we use the following lemma:

Lemma A.5: The nonsingular stabilizing solutions $X(\bar{\tau})$ and $Y(\bar{\tau})$ to Riccati equations (18) and (19) are real analytic functions on the set \mathbb{T} .

Proof: As $X(\bar{\tau})$ is nonsingular, we can rewrite the Riccati equation (18) as

$$\begin{aligned} & (\lambda I + A - B_2 \bar{E}_1^{-1} D_{12}^T M_\tau C_1) + B_1 M_\mu M_\tau^{-1} M_\mu B_1^T X(\bar{\tau}) \\ & - B_2 \bar{E}_1^{-1} B_2^T X(\bar{\tau}) \\ & = -X^{-1}(\bar{\tau}) \left[\begin{aligned} & (\lambda I + A - B_2 \bar{E}_1^{-1} D_{12}^T M_\tau C_1)^T + \\ & C_1^T (M_\tau - M_\tau D_{12} \bar{E}_1^{-1} D_{12}^T M_\tau) C_1 X^{-1}(\bar{\tau}) \end{aligned} \right] \\ & \quad \times X(\bar{\tau}). \end{aligned} \quad (60)$$

As $X(\bar{\tau})$ is a pseudo-positive definite stabilizing solution to the Riccati equation (18), it follows that the matrix $-(\lambda I + A - B_2 \bar{E}_1^{-1} D_{12}^T M_\tau C_1)^T - C_1^T (M_\tau - M_\tau D_{12} \bar{E}_1^{-1} D_{12}^T M_\tau) C_1 X^{-1}(\bar{\tau})$ is Hurwitz and hence the pair $(-(\lambda I + A - B_2 \bar{E}_1^{-1} D_{12}^T M_\tau C_1)^T, -C_1^T (M_\tau - M_\tau D_{12} \bar{E}_1^{-1} D_{12}^T M_\tau) C_1)$ is stabilizable.

The Riccati equation (18) can be written as

$$\begin{aligned} & X^{-1}(\bar{\tau}) (\lambda I + A - B_2 \bar{E}_1^{-1} D_{12}^T M_\tau C_1)^T \\ & + (\lambda I + A - B_2 \bar{E}_1^{-1} D_{12}^T M_\tau C_1) X^{-1}(\bar{\tau}) \\ & + (B_1 M_\mu M_\tau^{-1} M_\mu B_1^T - B_2 \bar{E}_1^{-1} B_2^T) \\ & + X^{-1}(\bar{\tau}) C_1^T (M_\tau - M_\tau D_{12} \bar{E}_1^{-1} D_{12}^T M_\tau) C_1 X^{-1}(\bar{\tau}) = 0. \end{aligned} \quad (61)$$

Substituting the matrices $-(\lambda I + A - B_2 \bar{E}_1^{-1} D_{12}^T M_\tau C_1)^T$, $C_1^T (M_\tau - M_\tau D_{12} \bar{E}_1^{-1} D_{12}^T M_\tau) C_1$ and $-B_1 M_\mu M_\tau^{-1} M_\mu B_1^T + B_2 \bar{E}_1^{-1} B_2^T$ into A , R and Q of Theorem 2 in [22], respectively, it follows that $X^{-1}(\bar{\tau})$ is the maximal solution for all solutions of the Riccati equation (61). Since $C_1^T (M_\tau - M_\tau D_{12} \bar{E}_1^{-1} D_{12}^T M_\tau) C_1 \geq 0$ and $-B_1 M_\mu M_\tau^{-1} M_\mu B_1^T + B_2 \bar{E}_1^{-1} B_2^T$ is Hermitian, Theorem 4.1 in [23] is applicable. Using Theorem 4.1 in [23] by substituting $-(\lambda I + A - B_2 \bar{E}_1^{-1} D_{12}^T M_\tau C_1)^T$, $C_1^T (M_\tau - M_\tau D_{12} \bar{E}_1^{-1} D_{12}^T M_\tau) C_1$ and $-B_1 M_\mu M_\tau^{-1} M_\mu B_1^T + B_2 \bar{E}_1^{-1} B_2^T$ into A , R and Q , respectively, gives $X^{-1}(\bar{\tau})$ is a real analytic function of $\bar{\tau} \in \mathbb{T}$. This further implies that $X(\bar{\tau})$ is a real analytic function of $\bar{\tau} \in \mathbb{T}$. Similarly, we can verify that $Y(\bar{\tau})$ is also a real analytic function of $\bar{\tau} \in \mathbb{T}$. \square

Proof of Theorem 5.2: Let $\varepsilon > 0$ be chosen to be sufficiently small so that the set $\mathbb{B}(\bar{\tau}, \varepsilon) = \{\bar{\tau} \in \mathcal{R}_+^m : \|\bar{\tau} - \bar{\tau}\| < \varepsilon\} \subset \{\bar{\tau} \in \mathbb{F} : \text{Condition I, II and III of Theorem 5.1 holds}\}$. The existence of such an $\varepsilon > 0$ follows from Lemma A.5.

Since $X(\bar{\tau})$ and $Y(\bar{\tau})$ are analytic function on the set \mathbb{T} , it straightforward to verify that $f(\bar{\tau})$ is an analytic function on the set \mathbb{F} . Since Δ^{-1} is a diagonal positive definite matrix, it follows that Condition II of the theorem implies that $\det J(\bar{\tau}) \neq 0$. Let $c = f(\bar{\tau})$. It follows from the Inverse Function Theorem (e.g., see Theorem 7.8 in [24]) that there is an open ball $\mathbb{B}(c, \iota)$ and a unique continuously differentiable function g from $\mathbb{B}(c, \iota)$ into $\mathbb{B}(\bar{\tau}, \varepsilon)$ such that $\bar{\tau} = g(c)$ and $f(g(\bar{c})) = \bar{c}$ for all $\bar{c} \in \mathbb{B}(c, \iota)$.

Since the set of rational vectors \mathcal{Q}^m is dense in \mathcal{R}^m , we can choose $\check{c} \in \mathbb{B}(c, \iota)$ such that all the elements of \check{c} are rational and non-zero. Also, it follows from the above discussion that there exists a point $\check{\tau} \in \mathbb{B}(\bar{\tau}, \varepsilon)$ such that $f(\check{\tau}) = \check{c}$ where $\check{\tau} = g(\check{c})$. Therefore, Condition IV of Theorem 5.1 is satisfied.

It follows from the definition of $\mathbb{B}(\bar{\tau}, \varepsilon)$ that $\check{\tau}$ satisfies Conditions I, II and III of Theorem 5.1. Hence, Theorem

5.1 implies that the corresponding closed-loop system is pendulum-like and Lagrange stable. \square

J. Proof of Theorem 6.1:

Substituting the controller law (29) into the system (5a), (5b) gives the closed-loop system

$$\begin{aligned} \dot{x} &= (A - B_2 \bar{E}_1^{-1} D_{12}^T M_\tau C_1 - B_2 \bar{E}_1^{-1} B_2^T X)x + B_1 \xi, \\ z &= ((I - D_{12} \bar{E}_1^{-1} D_{12}^T M_\tau) C_1 - D_{12} \bar{E}_1^{-1} B_2^T X)x. \end{aligned} \quad (62)$$

Since $T^{-1}(A - B_2 \bar{E}_1^{-1} D_{12}^T M_\tau C_1 - B_2 \bar{E}_1^{-1} B_2^T X)\bar{d} = (\bar{A} - \bar{B}_2 \bar{E}_1^{-1} D_{12}^T M_\tau \bar{C}_1 - \bar{B}_2 \bar{E}_1^{-1} \bar{B}_2^T \bar{X})\bar{d} = 0$, it follows that $(A - B_2 \bar{E}_1^{-1} D_{12}^T M_\tau C_1 - B_2 \bar{E}_1^{-1} B_2^T X)\bar{d} = 0$. Using this fact and condition II, it follows from Lemma 2.1 that the closed-loop system (62) is a pendulum-like system with respect to $\Pi(\bar{p}\tau_0\bar{d})$.

Using the fact that the Riccati equation (18) has a pseudo-positive definite solution and Condition I holds, Theorem 3.3 implies that the closed-loop system (62) is pseudo strict bounded real. Then, using Lemma 2.2, it follows that the closed-loop system (62), (5b), (2), (3), (4) is Lagrange stable. \square

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